Hash Proof Systems over Lattices Revisited

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Abstract. Hash Proof Systems or Smooth Projective Hash Functions (SPHFs) are a form of implicit arguments introduced by Cramer and Shoup at Eurocrypt'02. They have found many applications since then, in particular for authenticated key exchange or honest-verifier zero-knowledge proofs. While they are relatively well understood in group settings, they seem painful to construct directly in the lattice setting.

Only one construction of an SPHF over lattices has been proposed, by Katz and Vaikuntanathan at Asiacrypt'09. But this construction has an important drawback: it only works for an ad-hoc language of ciphertexts. Concretely, the corresponding decryption procedure needs to be tweaked, now requiring q many trapdoor inversion attempts, where q is the modulus of the underlying Learning With Error (LWE) problem.

Using harmonic analysis, we explain the source of this limitation, and propose a way around it. We show how to construct SPHFs for standard languages of LWE ciphertexts, and explicit our construction over a tag-IND-CCA2 encryption scheme à la Micciancio-Peikert (Eurocrypt'12). We then improve our construction and our analysis in the case where the tag is known in advance or fixed (in the latter case, the scheme is only IND-CPA) with a super-polynomial modulus, to get a stronger type of SPHF, which was never achieved before for any language over lattices.

Finally, we conclude with applications of these SPHFs: password-based authenticated key exchange, honest-verifier zero-knowledge proofs, and a relaxed version of witness encryption.

Keywords. Hash Proof Systems, SPHF, Lattices, Learning With Errors, Harmonic Analysis.

1 Introduction

Harmonic analysis is a powerful tool in geometry of numbers, especially in combination with Gaussian measure, which has lead to important progress on transference theory [Ban93]. Those tools also played a crucial role for the foundation of lattice-based cryptography, being at the heart of proofs of worst-case hardness for lattice problems, such as the Short Integer Solution problem (SIS) and the Learning with Errors (LWE) problem [MR04, Reg05, GPV08]. Later, security proofs relied on a few convenient lemmas in a black-box manner, and for most applications this was sufficient: lattice-based cryptography quickly caught up with pairing-based cryptography, for example with the constructions of (Hierarchical) Identity Based Encryption's [GPV08, CHKP10, MP12] and beyond [Boy13, GVW13, GVW15].

There nevertheless remains one primitive for which lattice-based cryptography is still far behind: Hash Proof Sytems or Smooth Projective Hash Functions (SPHFs) [CS02]. Beyond the original Chosen-Ciphertext secure encryption scheme of Cramer and Shoup [CS98], SPHFs give rise to generalized classes of Authenticated Key Exchange (Password-based, Language-based, ...) [GL06,

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ACP09, KV11, BBC+13a]. They also have been used in Oblivious Transfer [Kal05, ABB+13], One-Time Relatively-Sound Non-Interactive Zero-Knowledge Arguments [JR12], and Zero-Knowledge Arguments [BBC+13b].

An SPHF can be seen as an implicit (designated-verifier) zero-knowledge proof for a language. The most useful languages for SPHFs are the languages of ciphertexts of a given plaintext M.

To our knowledge, there is only one construction of SPHF for a lattice-based encryption scheme, given by Katz and Vaikuntanathan [KV09], and no subsequent work.[5] However, their construction has a main drawback: the language of their SPHF is not simply defined as the set of valid standard LWE ciphertexts. Naturally, the set of valid ciphertexts of 0 should correspond to the set of ciphertexts close to the lattice defined by the public key. Instead, their language includes all the ciphertexts $\boldsymbol{c}$ such that at least one integer multiple is close to the public lattice. This makes the decryption procedure very costly (about q trapdoor inversions), and forbids the use of super-polynomial modulus q. This limitation is a serious obstacle to the construction of a stronger type of SPHF, called KV-SPHF in reference to [KV11], for which the projection key (which can be seen as the public key of the SPHF) does not depend on the ciphertext $\boldsymbol{c}$.

This strongly contrasts with SPHFs in a group-based setting, which can handle classical ElGamal or Cramer-Shoup encryption schemes —for example [CS02, GL06]— without any modification of the decryption procedure. This is a technical hassle to carry when building on top of such an SPHF.

We therefore view as an important question to determine whether this caveat is inherent to lattice-based SPHFs, or if it can be overcome. We shall find an answer by re-introducing some harmonic analysis.

1.1 Contributions

Our main contribution consists in constructing SPHFs for standard lattice-based encryption schemes. We provide general theorems to ease the proofs of correctness and security (a.k.a., smoothness or universality) of SPHFs over standard lattice-based encryption schemes. We detail two particular instantiations: one over an IND-CCA2 encryption scheme à la Micciancio-Peikert [MP12], and one over a IND-CPA restriction of the same scheme. While the second instantiation is over a simpler language, it is a stronger type of SPHF, namely it is a KV-SPHF. To our knowledge, this is the first KV-SPHF over any lattice-based language.

As with many zero-knowledge-type primitives in the lattice setting [Lyu08, Lyu09] and as with the SPHF of [KV09], there is a gap between the correctness property and the security property. Concretely, smoothness holds for ciphertexts which do not decrypt to a given message, while correctness holds only for honestly generated ciphertexts. However, contrary to [KV09], we use a standard encryption scheme and do not need to tweak the decryption procedure nor the language. We thus avoid the main caveat of the latter paper.

More precisely, using harmonic analysis we explain the reason for the caveat of the SPHF of [KV09], namely the presence of many harmonics in the q-periodic function used to extract entropy from the approximately shared secret (this extracting function being the usual deterministic rounding function, corresponding to a square signal, in [KV09]). According to our Theorem 3.2, we can guarantee statistical smoothness for invalid ciphertexts using one decryption attempt per couple of conjugate harmonics (seen as complex functions) of the entropy extracting function.

Having identified the source of the caveat, it becomes clear how to repair it: the entropy extracting function should be randomized, with a weight following a *pure cosine*. This decreases the number of harmonics to three (the average and one pair of complex conjugates), and therefore the number of required decryptions to one (Corollary 3.3 and Theorem 3.5).

This solution nevertheless does only provide approximate correctness, which is also problematic for some applications. This can be solved using correctness amplification via codes, but at the price of preventing the resulting SPHF to be a KV-SPHF.

[5] Except for a retracted draft by Blazy et al. [BCDP13].

In our second instantiation, we therefore proceed to construct an almost-square rounding function, which offers statistical correctness[6] and imperfect universality, (namely $(1/3 + o(1))$-universality, as proved in Theorem 4.5). This instantiation requires a more subtle analysis, taking account of *destructive interferences*. We then can amplify universality to get statistical smoothness while keeping a statistical correctness. Contrary to the correctness amplification, this transformation preserves the independence of the projection key from the ciphertext. In particular, if the ciphertexts are from an IND-CPA scheme à la Micciancio-Peikert, then we get the first KV-SPHF over a lattice-based language.

This KV-SPHF uses a *super-polynomial modulus* q. It seems hard to construct such a KV-SPHF for a polynomial modulus, as a KV-SPHF for an IND-CPA encryption scheme directly yields a one-round key exchange (where each party sends a ciphertext of 0 and a projection key, and where the resulting session key is the xor of the two corresponding hash values) and we do not know of any lattice-based one-round key exchange using a polynomial modulus.

Having built these new SPHFs, we can now proceed with several applications showing that the gap between smoothness (or universality) and correctness is not an issue in most cases. We start by proposing an efficient password-authenticated key exchange (PAKE) scheme in three flows. We do so by plugging our first SPHF in the framework from [KV09]. Using in addition our KV-SPHF and following the GL-PAKE construction from [ABP15b] which is an improvement of the Gennaro-Lindell framework [KOY01, GL06], we get the first Gennaro-Lindell-based PAKE in two flows over lattices.[7]

We also show how to construct honest-verifier zero-knowledge proofs for any NP language from lattice-based SPHF. We conclude by showing a relaxed version of witness encryption for some lattice-based languages. Witness encryption is a very recent primitive introduced in [GGSW13] which enables a user to encrypt a message to a given word of some NP language. The message can be decrypted using a witness for the word.

1.2 Open Question

We see as the main open question to extend our techniques to their full extent in the ring-setting. More precisely, our SPHF only produces one bit, and is easily extended to the ring-setting still asking for 1 bit. This requires costly repetitions for applications, and one would hope that a ring setting variant could directly produce $\Theta(n)$ bits.

1.3 Road Map

We start by some preliminaries on lattices and SPHFs in Section 2. In particular, we define several variants of lattice-based (approximate) SPHFs (in particular universal bit-PHFs) and formally show various transformations which were only implicit in [KV09].

In Section 3, we then show step-by-step how to construct an SPHF for IND-CCA2 ciphertexts à la Micciancio-Peikert and how to avoid the caveat of the construction of [KV09].

In Section 4, we construct a KV-SPHF for ciphertexts under a IND-CPA scheme à la Micciancio-Peikert, when the modulus is super-polynomial.

In Section 5, we conclude by exhibiting several applications.

Figure 1 summarizes our results and the paper road map. All the notions in this figure are formally defined in Section 2.

[6] More precisely, the probability of error is $\mathrm{poly}(n, \sigma)/q$, which is $\mathrm{negl}(n)$ for super-polynomial approximation factors q/σ.

[7] We should point out however, that it is also possible to construct a 2-round PAKE by combining [KV09] and [GK10] (a generalization of [JG04]). But the resulting PAKE would not follow the framework of Gennaro and Lindell [GL06].

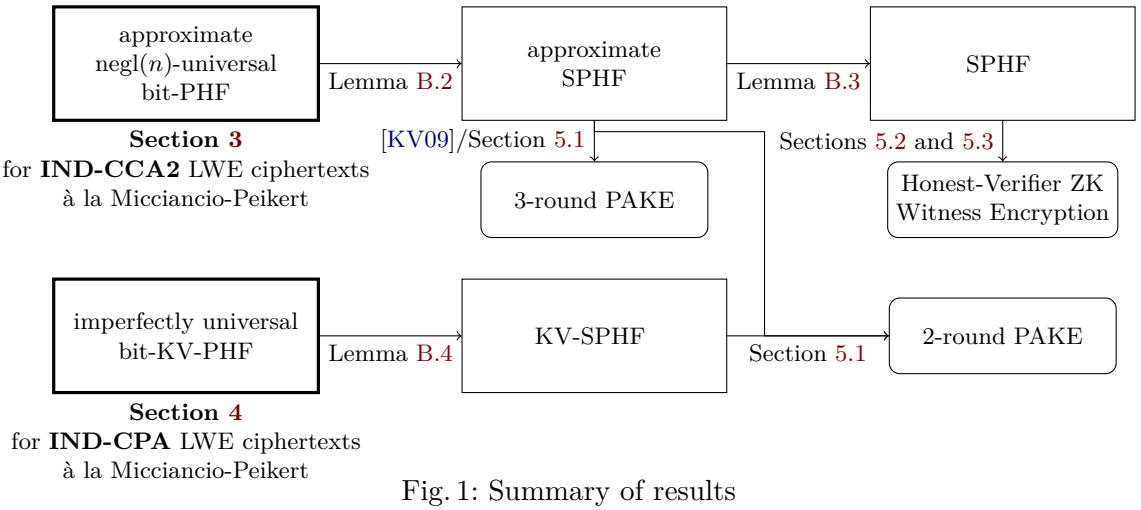


Fig. 1: Summary of results

2 Preliminaries

2.1 Notations

The security parameter is denoted n. The notation $\mathrm{negl}(n)$ denotes any function f such that $f(n) = n^{-\omega(1)}$. For a probabilistic algorithm $\mathsf{alg}(\mathsf{inputs})$, we may explicit the randomness it uses with the notation $\mathsf{alg}(\mathsf{inputs}\,;\,\mathsf{coins})$, otherwise the random coins are implicitly fresh.

Column vectors will be denoted by bold lower-case letters, e.g. $\boldsymbol{x}$, and matrices will be denoted by bold upper-case letters, e.g. $\boldsymbol{A}$. If $\boldsymbol{x}$ is vector and $\boldsymbol{A}$ is a matrix, $\boldsymbol{x}^t$ and $\boldsymbol{A}^t$ will denote their transpose. We use $[\boldsymbol{A}|\boldsymbol{B}]$ for the horizontal concatenation of matrices, and $[\boldsymbol{A}\,;\,\boldsymbol{B}] = [\boldsymbol{A}^t|\boldsymbol{B}^t]^t$ for the vertical concatenation. For $\boldsymbol{x} \in \mathbb{R}^m$, $\|\boldsymbol{x}\|$ will denote the canonical euclidean norm of $\boldsymbol{x}$. We will use $\mathcal{B}$ to denote the euclidean ball of radius 1, where, unless specifically stated otherwise, the ball is m-dimensional. If $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ will denote their canonical inner product, and $d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|$ their distance. If $E \subset \mathbb{R}^m$ is countable and discrete, we will denote $d(\boldsymbol{x}, E) = \min_{\boldsymbol{y} \in E} d(\boldsymbol{x}, \boldsymbol{y})$. For a function $f\colon E \to \mathbb{C}$ or $f\colon E \to \mathbb{R}$, $f(E)$ will denote the sum $\sum_{\boldsymbol{x} \in E} f(x)$. For $a, b \in \mathbb{R}$, $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ will denote the closed real interval with endpoints a and b, $\lfloor a \rfloor$, $\lceil a \rceil$, and $\lfloor a \rceil$ will respectively denote the largest integer smaller than a, the smallest integer greater than a, and the closest integer to a (the largest one if there are two). The xor of two bit strings $a, b \in \{0,1\}^k$ is denoted by $a \oplus b$.

The modulus $q \in \mathbb{Z}$ will be taken as an odd prime, for simplicity.

2.2 Lattices and Gaussians

Lattices. An m-dimensional *lattice* Λ is a discrete subgroup of $\mathbb{R}^m$. Equivalently, Λ is a lattice if it can be written $\Lambda = \{\boldsymbol{B}\boldsymbol{s} \mid \boldsymbol{s} \in \mathbb{Z}^n\}$ where $n \leq m$, for some $\boldsymbol{B} \in \mathbb{R}^{m \times n}$, where the columns of $\boldsymbol{B}$ are linearly independent. In that case, $\boldsymbol{B}$ is called a *basis* of Λ. Then, we define the *determinant* of Λ as $\det(\Lambda) = \sqrt{\det(\boldsymbol{B}^t\boldsymbol{B})}$, which does not depend on the choice of the basis $\boldsymbol{B}$.

We define the *dual lattice* of Λ as

$$\Lambda^* = \{\boldsymbol{x} \in \mathrm{Span}_{\mathbb{R}}(\Lambda) \mid \forall \boldsymbol{y} \in \Lambda, \langle \boldsymbol{x}, \boldsymbol{y} \rangle \in \mathbb{Z}\} \ .$$

Recall the identity $(\Lambda^*)^* = \Lambda$. Given $\boldsymbol{A} \in \mathbb{Z}_q^{m \times n}$ where $m \geq n$, and modulus $q \geq 2$, we define the following q-ary lattices

$$\Lambda(\boldsymbol{A}) = \{\boldsymbol{A}\boldsymbol{s} \mid \boldsymbol{s} \in \mathbb{Z}_q^n\} + q\mathbb{Z}^m \ , \qquad \Lambda^\perp(\boldsymbol{A}) = \{\boldsymbol{h} \in \mathbb{Z}^m \mid \boldsymbol{h}^t\boldsymbol{A} = \boldsymbol{0}^t \bmod q\} \ .$$

Note that up to a scaling factor, $\Lambda(\boldsymbol{A})$ and $\Lambda^\perp(\boldsymbol{A})$ are dual of each other: $\Lambda(\boldsymbol{A}) = q \cdot \Lambda^\perp(\boldsymbol{A})^*$. For a syndrome $\boldsymbol{p} \in \mathbb{Z}_q^n$, we define the coset of $\Lambda^\perp(\boldsymbol{A})$:

$$\Lambda_{\boldsymbol{p}}^\perp(\boldsymbol{A}) = \{\boldsymbol{h} \in \mathbb{Z}^m \mid \boldsymbol{h}^t\boldsymbol{A} = \boldsymbol{p}^t \bmod q\} \ .$$

When there is no confusion about which matrix $\boldsymbol{A}$ is used, we will simply denote them Λ, $\Lambda^\perp$, and $\Lambda_{\boldsymbol{p}}^\perp$ respectively.

Gaussians. If $s > 0$ and $\boldsymbol{c} \in \mathbb{R}^m$, we define the *Gaussian weight function* on $\mathbb{R}^m$ as

$$\rho_{s,\boldsymbol{c}} \colon \boldsymbol{x} \mapsto \exp(-\pi \|\boldsymbol{x} - \boldsymbol{c}\|^2 / s^2).$$

Similarly, if Λ is an m-dimensional lattice, we define the *discrete Gaussian distribution* over Λ, of parameter s and centered in $\boldsymbol{c}$ by:

$$\forall \boldsymbol{x} \in \Lambda,\ D_{\Lambda,s,\boldsymbol{c}}(\boldsymbol{x}) = \frac{\rho_{s,\boldsymbol{c}}(x)}{\rho_{s,\boldsymbol{c}}(\Lambda)} \ .$$

When $\boldsymbol{c} = \boldsymbol{0}$, we will simply write ρ_s and $D_{\Lambda,s}$. We recall the tail-bound of Banaszczyk for discrete Gaussians:

Lemma 2.1 ([Ban93, Lemma 1.5], as stated in [MR04, Lemma 2.10]). *For any $c > 1/\sqrt{2\pi}$, m-dimensional lattice Λ and any vector $\boldsymbol{v} \in \mathbb{R}^m$:*

$$\rho_s(\Lambda \setminus sc\sqrt{m}\mathcal{B}) \leq C^m \rho_s(\Lambda) \ , \qquad \rho_s((\Lambda + \boldsymbol{v}) \setminus sc\sqrt{m}\mathcal{B}) \leq 2C^m \rho_s(\Lambda) \ .$$

where $C = c\sqrt{2\pi e} \cdot e^{-\pi c^2} < 1$.

An important quantity associated to a lattice is its *smoothing parameter*, introduced by Micciancio and Regev [MR04]:

Definition 2.2 (Smoothing parameter [MR04]). *For $\epsilon > 0$, the smoothing parameter of a lattice Λ, denoted $\eta_\epsilon(\Lambda)$, is the smallest $s > 0$ such that $\rho_{1/s}(\Lambda^* \setminus \{0\}) \leq \epsilon$.*

The following lemma states that if the parameter of the discrete Gaussian is above the smoothing parameter of the lattice, then the Gaussian weight of the cosets of Λ are essentially the same:

Lemma 2.3 ([Reg05, Claim 3.8]). *For any lattice $\Lambda \subset \mathbb{R}^m$, $\boldsymbol{c} \in \mathbb{R}^m$, and $s \geq \eta_\epsilon(\Lambda)$:*

$$(1-\epsilon)s^m \det(\Lambda^*) \leq \rho_s(\Lambda + \boldsymbol{c}) \leq (1+\epsilon)s^m \det(\Lambda^*) \ .$$

The smoothing parameter of the dual of a random q-ary lattice can be controlled using the following:

Lemma 2.4 (Corollary of [MP12, Lemma 2.4]). *Fix parameters n, q a prime, and $m \geq \Theta(n \log q)$. Let $\epsilon \geq 2^{-O(n)}$ and $s > 2\eta_\epsilon(\mathbb{Z}^m)$. Fix $0 < \delta \leq 1$. Then, for $\boldsymbol{A}$ uniformly random in $\mathbb{Z}_q^{m \times n}$, we have $s \geq \eta_{2\epsilon/\delta}(\Lambda^\perp(\boldsymbol{A}))$ except with probability at most δ over the choice of $\boldsymbol{A}$.*

To instantiate the above, we recall the smoothing parameter of $\mathbb{Z}^m$.

Lemma 2.5 (Corollary of [MR04, Lemma 3.3]). *For all integer $m \geq 1$, $\epsilon \in (0, 1/2)$, the smoothing parameter of $\mathbb{Z}^m$ satisfies $\eta_\epsilon(\mathbb{Z}^m) \leq C\sqrt{\log(m/\epsilon)}$ for some universal constant $C > 0$.*

Harmonic analysis. Let us recall the exponential basis of periodic functions and their vectorial analogues:

$$e_x \colon y \mapsto \exp(2i\pi xy) \ , \qquad e_{\boldsymbol{x}} \colon \boldsymbol{y} \mapsto \exp(2i\pi \langle \boldsymbol{x}, \boldsymbol{y} \rangle) \ .$$

The Fourier transform of $f : \mathbb{R}^m \to \mathbb{C}$ is defined by:

$$\hat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^m} f(\boldsymbol{x}) e^{-2i\pi\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} d\boldsymbol{x} \ .$$

The Fourier transform of the Gaussian weight function ρ_s is $\widehat{\rho_s} = s^m \rho_{1/s}$. Recall the time-shift-phase-shift identity: if $g(\boldsymbol{x}) = f(\boldsymbol{x}) e_{\boldsymbol{z}}(\boldsymbol{x})$ for some $\boldsymbol{z} \in \mathbb{R}^m$, then $\hat{g}(\boldsymbol{\xi}) = \hat{f}(\boldsymbol{\xi} - \boldsymbol{z})$. Similarly, if $g(\boldsymbol{x}) = f(\boldsymbol{x} + \boldsymbol{t})$ for some $\boldsymbol{t} \in \mathbb{R}^m$, then $\hat{g}(\boldsymbol{\xi}) = \hat{f}(\boldsymbol{\xi}) e_{\boldsymbol{t}}(\boldsymbol{\xi})$. For two functions $f, g : \mathbb{R}^m \to \mathbb{C}$, we will denote by $f \odot g$ their convolution product:

$$f \odot g(\boldsymbol{x}) = \int_{\mathbb{R}^m} f(\boldsymbol{y}) g(\boldsymbol{x} - \boldsymbol{y}) d\boldsymbol{y} \ .$$

The Fourier transform turns convolutions into pointwise products, and conversely:

$$\widehat{f \odot g}(\boldsymbol{\xi}) = \hat{f}(\boldsymbol{\xi}) \cdot \hat{g}(\boldsymbol{\xi}) \ , \qquad \widehat{f \cdot g}(\boldsymbol{\xi}) = \hat{f}(\boldsymbol{\xi}) \odot \hat{g}(\boldsymbol{\xi}) \ .$$

Finally, let us recall the Poisson summation formula:

Lemma 2.6 (Poisson summation formula). *For any lattice Λ and $f : \mathbb{R}^m \to \mathbb{C}$, we have:*

$$f(\Lambda) = \det(\Lambda^*) \hat{f}(\Lambda^*) \ .$$

Learning with Errors.

Definition 2.7 (Learning with Errors (LWE)). *Let $q \geq 2$, and χ be a distribution over $\mathbb{Z}$. The Learning with Errors problem $\mathrm{LWE}_{\chi,q}$ consists in, given polynomially many samples, distinguishing the two following distributions:*

- *$(\boldsymbol{a}, \langle \boldsymbol{a}, \boldsymbol{s} \rangle + e)$, where $\boldsymbol{a}$ is uniform in $\mathbb{Z}_q^n$, $\boldsymbol{e} \leftarrow \chi$, and $\boldsymbol{s} \in \mathbb{Z}_q^n$ is a fixed secret chosen uniformly,*
- *$(\boldsymbol{a}, b)$, where $\boldsymbol{a}$ is uniform in $\mathbb{Z}_q^n$, and b is uniform in $\mathbb{Z}_q$.*

In [Reg05], Regev showed that for $\chi = D_{\mathbb{Z},\sigma}$, for any $\sigma \geq 2\sqrt{n}$, and q such that $q/\sigma = \mathrm{poly}(n)$, $\mathrm{LWE}_{\chi,q}$ is at least as hard as solving worst-case SIVP for polynomial approximation factors.

Trapdoor for LWE. Throughout this paper, we will use the trapdoors introduced in [MP12] to build our public matrix $\boldsymbol{A}$. Define $g_{\boldsymbol{A}}(\boldsymbol{s}, \boldsymbol{e}) = \boldsymbol{A}\boldsymbol{s} + \boldsymbol{e}$, let $\boldsymbol{G}^t = \boldsymbol{I}_n \otimes \boldsymbol{g}^t$, where $\boldsymbol{g}^t = [1, 2, \ldots, 2^k]$ and $k = \lceil \log q \rceil - 1$, and let $\boldsymbol{H} \in \mathbb{Z}_q^{n \times n}$ be invertible.

Lemma 2.8 ([MP12, Theorems 5.1 and 5.4]). *There exist two PPT algorithms $\mathsf{TrapGen}$ and $g_{(\cdot)}^{-1}$ with the following properties assuming $q \geq 2$ and $m \geq \Theta(m \log q)$:*

- *$\mathsf{TrapGen}(1^n, 1^m, q)$ outputs $(\boldsymbol{T}, \boldsymbol{A}_0)$, where the distribution of the matrix $\boldsymbol{A}_0$ is at negligible statistical distance from uniform in $\mathbb{Z}_q^{m \times n}$, and such that $\boldsymbol{T}\boldsymbol{A}_0 = \boldsymbol{0}$, where $s_1(\boldsymbol{T}) \leq O(\sqrt{m})$ and where $s_1(\boldsymbol{T})$ is the operator norm of $\boldsymbol{T}$, which is defined as $\max_{\boldsymbol{x} \neq 0} \|\boldsymbol{T}\boldsymbol{x}\| / \|\boldsymbol{x}\|$.*[8]
- *Let $(\boldsymbol{T}, \boldsymbol{A}_0) \leftarrow \mathsf{TrapGen}(1^n, 1^m, q)$. Let $\boldsymbol{A}_{\boldsymbol{H}} = \boldsymbol{A}_0 + [\boldsymbol{0}\,;\, \boldsymbol{G}\boldsymbol{H}]$ for some invertible matrix $\boldsymbol{H}$ called a* tag. *Then, we have $\boldsymbol{T}\boldsymbol{A} = \boldsymbol{G}\boldsymbol{H}$. Furthermore, if $\boldsymbol{x} \in \mathbb{Z}_q^m$ can be written as $\boldsymbol{A}_{\boldsymbol{H}}\boldsymbol{s} + \boldsymbol{e}$ where $\|\boldsymbol{e}\| \leq B' := q/\Theta(\sqrt{m})$, then $g_{\boldsymbol{A}_{\boldsymbol{H}}}^{-1}(\boldsymbol{T}, \boldsymbol{x}, \boldsymbol{H})$ outputs $(\boldsymbol{s}, \boldsymbol{e})$.*

We will simply write $g_{\boldsymbol{A}}^{-1}(\boldsymbol{T}, \boldsymbol{x})$ when $\boldsymbol{H} = \boldsymbol{I}_n$.

More precisely, to sample $(\boldsymbol{T}, \boldsymbol{A}_0)$ with $\mathsf{TrapGen}$, we sample a uniform $\bar{\boldsymbol{A}} \in \mathbb{Z}_q^{\bar{m} \times n}$ where $\bar{m} = m - nk = \Theta(n \log q)$, and some $\boldsymbol{R} \leftarrow \mathcal{D}^{nk \times \bar{m}}$, where the distribution $\mathcal{D}^{nk \times \bar{m}}$ assigns probability $1/2$ to 0, and $1/4$ to ± 1. We output $\boldsymbol{T} = [-\boldsymbol{R} \,|\, \boldsymbol{I}_{nk}]$ along with $\boldsymbol{A}_0 = [\bar{\boldsymbol{A}}\,;\, \boldsymbol{R}\bar{\boldsymbol{A}}]$. Then, given a tag $\boldsymbol{H}$, we have: $\boldsymbol{T}(\boldsymbol{A}_0 + [\boldsymbol{0}\,;\, \boldsymbol{G}\boldsymbol{H}]) = \boldsymbol{G}\boldsymbol{H}$.

[8] The bound on $s_1(\boldsymbol{T})$ holds except with probability at most 2^{-n} in the original construction, but for convenience we assume the algorithm restarts if it does not hold.

Tag-IND-CCA2 LWE encryption à la Micciancio-Peikert. For our applications, we will need a (labelled) encryption scheme that is IND-CCA2 (the definition is given in Appendix A.1). This can be built generically and efficiently from a tag-IND-CCA2 encryption scheme, as recalled in Appendix A.2. Below, we describe a simplified variant of the scheme of [MP12, Sec. 6.3].

For this scheme, we assume q to be an odd prime. We set an encoding function for messages $\mathsf{Encode}(\mu \in \{0,1\}) = \mu \cdot (0, \dots 0, \lceil q/2 \rceil)^t$. Note that $2 \cdot \mathsf{Encode}(\mu) = (0, \dots, 0, \mu) \bmod q$.

Let $\mathcal{R}$ be a ring with a subset $\mathcal{U} \subset \mathcal{R}^\times$ of invertible elements, of size 2^n, and with the *unit differences* property: if $u_1 \neq u_2 \in \mathcal{U}$, then $u_1 - u_2$ is invertible in $\mathcal{R}$. Let h be an injective ring homomorphism from $\mathcal{R}$ to $\mathbb{Z}_q^{n \times n}$ (see [MP12, Section 6.1 and 6.3] for an explicit construction). Note that if $u_1 \neq u_2 \in \mathcal{U}$, then $h(u_1 - u_2)$ is invertible, and thus an appropriate tag $H = h(u_1 - u_2)$ for the trapdoor.

Let $(\boldsymbol{T}, \boldsymbol{A}_0) \leftarrow \mathsf{TrapGen}(1^n, 1^m, q)$. The public encryption key is $\mathsf{ek} = \boldsymbol{A}_0$, and the secret decryption key is $\mathsf{dk} = \boldsymbol{T}$.

- $\mathsf{Encrypt}(\mathsf{ek} = \boldsymbol{A}_0,\ u \in \mathcal{U},\ \mu \in \{0,1\})$ encrypts the message μ under the public key ek and for the tag u, as follows: Let $\boldsymbol{A}_u = \boldsymbol{A}_0 + [\boldsymbol{0}\,;\, \boldsymbol{G}h(u)]$. Pick $\boldsymbol{s} \in \mathbb{Z}_q^n$, $\boldsymbol{e} \leftarrow D_{\mathbb{Z},t}^m$ where $t = \sigma\sqrt{m} \cdot \omega(\sqrt{\log n})$. Restart[9] if $\|\boldsymbol{e}\| > B$, where $B \coloneqq 2t\sqrt{m}$. Output the ciphertext:
$$\boldsymbol{c} = \boldsymbol{A}_u \boldsymbol{s} + \boldsymbol{e} + \mathsf{Encode}(\mu) \bmod q \ .$$
- $\mathsf{Decrypt}(\mathsf{dk} = \boldsymbol{T},\ u \in \mathcal{U},\ \boldsymbol{c} \in \mathbb{Z}_q^m)$ decrypts the ciphertext $\boldsymbol{c}$ for the tag u using the decryption key dk as follows: Output
$$\begin{cases} \mu & \text{if } g_{\boldsymbol{A}_u}^{-1}(\boldsymbol{T}, 2\boldsymbol{c}, h(u)) = 2\boldsymbol{e} + (0, \dots, 0, \mu) \text{ where } \boldsymbol{e} \in \mathbb{Z}^m \text{ and } \|\boldsymbol{e}\| \leq B' \ , \\ \bot & \text{otherwise.}[10] \end{cases}$$

Since $\lceil q/2 \rceil$ is the inverse of 2 mod q, we have
$$\mu' \coloneqq \mathsf{Decrypt}(\boldsymbol{T}, u, \boldsymbol{c}) \neq \bot \quad \iff \quad d(\boldsymbol{c} - \mathsf{Encode}(\mu'), \Lambda(\boldsymbol{A}_u)) < B' \ .$$

Suppose that $m \geq \theta(n \log q)$. Note that $d(\mathsf{Encode}(1), \Lambda(\boldsymbol{A}_u)) > B'$ simultaneously for all u with overwhelming probability over the randomness of $\mathsf{TrapGen}$ (using a union bound, as in [GPV08, Lemma 5.3] for instance). Then, by Lemma 2.8, the scheme is correct as long as $B \leq B'$, or equivalently

$\sigma m^{3/2} \cdot \omega(\sqrt{\log n}) \leq q$.

Theorem 2.9. *Assume $m \geq \Theta(n \log q)$. The above scheme is tag-IND-CCA2 assuming the hardness of the $\mathrm{LWE}_{\chi,q}$ problem for $\chi = D_{\mathbb{Z},\sigma}$.*

The precise definition for tag-IND-CCA2 is detailed in Appendix A.1, and the proof is given in Appendix A.3.

Remark 2.10. If a constant tag u is hardcoded in $\mathsf{Encrypt}$ and $\mathsf{Decrypt}$, then the resulting encryption scheme is just an IND-CPA scheme using trapdoors from [MP12].

Lemma 2.11. *Assume $m \geq \Theta(n \log q)$. With $\boldsymbol{A}_0$ sampled as above, except with probability 2^{-n}, it holds that*
$$\forall u \in \mathcal{U}, \quad \eta_{2^{-n}}(\Lambda^\perp(\boldsymbol{A}_u)) \leq C\sqrt{n}$$
for some universal constant C.

Proof. Note that $\boldsymbol{A}_0$ is (about) uniform under the randomness of $\mathsf{TrapGen}$, and so is $\boldsymbol{A}_u$ for a fixed $u \in \mathcal{U}$. Apply Lemma 2.4 and Lemma 2.5 with $\epsilon = 8^{-n}/2$ and $\delta = 4^{-n}$ to $\boldsymbol{A}_u$, ensuring that $\eta_{2^{-n}}(\Lambda^\perp(\boldsymbol{A}_u)) \leq C\sqrt{n}$ except with probability δ. Conclude by the union bound over the 2^n elements $u \in \mathcal{U}$. □

[9] This happens only with exponentially small probability $2^{-\Theta(n)}$ by Lemma 2.1.

[10] Note that the inversion algorithm $g_{(\cdot)}^{-1}$ can succeed even if $\|\boldsymbol{e}\| > B'$, depending on the randomness of the trapdoor. It is crucial to reject decryption nevertheless when $\|\boldsymbol{e}\| > B'$ to ensure CCA2 security. We also recall that $B' \coloneqq q/\Theta(\sqrt{m})$.

2.3 Approximate Smooth Projective Hash Functions

We consider approximate smooth projective hash functions (approximate SPHFs) defined in [KV09].

Languages. We consider a family of languages $(\mathscr{L}_{\mathsf{lpar},\mathsf{ltrap}})_{\mathsf{lpar},\mathsf{ltrap}}$ indexed by some *parameter* lpar and some *trapdoor* ltrap, together with a family of NP languages $(\bar{\mathscr{L}}_{\mathsf{lpar}})_{\mathsf{lpar}}$ indexed by some parameter lpar, with witness relation $\bar{\mathscr{R}}_{\mathsf{lpar}}$, such that:

$$\bar{\mathscr{L}}_{\mathsf{lpar}} = \{\chi \in \mathcal{X}_{\mathsf{lpar}} \mid \exists w,\ \bar{\mathscr{R}}_{\mathsf{lpar}}(\chi, w) = 1\} \ \subseteq\ \mathscr{L}_{\mathsf{lpar},\mathsf{ltrap}} \ \subseteq\ \mathcal{X}_{\mathsf{lpar}}\,,$$

where $(\mathcal{X}_{\mathsf{lpar}})_{\mathsf{lpar}}$ is a family of sets. The trapdoor ltrap and the parameter lpar are generated by a polynomial-time algorithm $\mathsf{Setup.lpar}$ which takes as input a unary representation of the security parameter n. We suppose that membership in $\mathcal{X}_{\mathsf{lpar}}$ and $\bar{\mathscr{R}}_{\mathsf{lpar}}$ can be checked in polynomial time given lpar and that membership in $\mathscr{L}_{\mathsf{lpar},\mathsf{ltrap}}$ can be checked in polynomial time given lpar and ltrap. The parameters lpar and ltrap are often omitted when they are clear from context.

We are mostly interested in languages of ciphertexts.

Example 2.12 (Languages of Ciphertexts). Let $(\mathsf{KeyGen}, \mathsf{Encrypt}, \mathsf{Decrypt})$ be a labeled encryption scheme. We define the following languages ($\mathsf{Setup.lpar} = \mathsf{KeyGen}$ and $(\mathsf{ltrap}, \mathsf{lpar}) = (\mathsf{dk}, \mathsf{ek})$):

$$\bar{\mathscr{L}} = \{(\mathsf{label}, C, M) \mid \exists \rho,\ C = \mathsf{Encrypt}(\mathsf{ek}, \mathsf{label}, M; \rho)\}\ ,$$
$$\mathscr{L} = \{(\mathsf{label}, C, M) \mid \mathsf{Decrypt}(\mathsf{dk}, \mathsf{label}, C) = M\}\ ,$$

where the witness relation $\bar{\mathscr{R}}$ is implicitly defined as:

$$\bar{\mathscr{R}}((\mathsf{label}, C, M), \rho) = 1 \quad\Longleftrightarrow\quad C = \mathsf{Encrypt}(\mathsf{ek}, \mathsf{label}, M; \rho)\ .$$

Approximate SPHFs. Let us now define approximate SPHFs following [KV09].

Definition 2.13. *Let $(\bar{\mathscr{L}}_{\mathsf{lpar}} \subseteq \mathscr{L}_{\mathsf{lpar},\mathsf{ltrap}} \subseteq \mathcal{X}_{\mathsf{lpar}})_{\mathsf{lpar},\mathsf{ltrap}}$ be languages defined as above. An approximate smooth projective hash function (SPHF) for these languages is defined by four probabilistic polynomial-time algorithms $(\mathsf{HashKG}, \mathsf{ProjKG}, \mathsf{Hash}, \mathsf{ProjHash})$:*

- $\mathsf{HashKG}(\mathsf{lpar})$ *generates a hashing key* hk *for the language parameters* lpar*;*
- $\mathsf{ProjKG}(\mathsf{hk}, \mathsf{lpar}, \chi)$ *derives a projection key* hp *from the hashing key* hk*, the language parameters* lpar*, and the word* χ*;*
- $\mathsf{Hash}(\mathsf{hk}, \mathsf{lpar}, \chi)$ *outputs a hash value* $\mathsf{H} \in \{0,1\}^\nu$ *(for some positive integer $\nu = \Omega(n)$) from the hashing key* hk*, for the word $\chi \in \mathcal{X}_{\mathsf{lpar}}$ and the language parameters* lpar*;*
- $\mathsf{ProjHash}(\mathsf{hp}, \mathsf{lpar}, \chi, w)$ *outputs a projected hash value* $\mathsf{pH} \in \{0,1\}^\nu$ *from the projection key* hp*, and the witness w, for the word $\chi \in \bar{\mathscr{L}}_{\mathsf{lpar}}$ (i.e., $\bar{\mathscr{R}}_{\mathsf{lpar}}(\chi, w) = 1$) and the language parameters* lpar*;*

which satisfy the following properties:

- ***Approximate correctness.*** *For any positive integer n, if $(\mathsf{ltrap}, \mathsf{lpar}) \leftarrow \mathsf{Setup.lpar}(1^n)$, with overwhelming probability over the randomness of* $\mathsf{Setup.lpar}$*, for any $\chi \in \bar{\mathscr{L}}_{\mathsf{lpar},\mathsf{ltrap}}$ (and associated witness w), the value* H *output by* $\mathsf{Hash}(\mathsf{hk}, \mathsf{lpar}, \chi)$ *is approximately determined by* $\mathsf{ProjKG}(\mathsf{hk}, \mathsf{lpar}, \chi)$ *relative to the Hamming metric. More precisely, writing $\mathsf{HW}(a, b)$ the Hamming distance between two strings $a, b \in \{0,1\}^\nu$, the SPHF is ϵ-correct, if:*

$$\Pr_{\mathsf{hk}}[\mathsf{HW}(\mathsf{Hash}(\mathsf{hk}, \mathsf{lpar}, \chi), \mathsf{ProjHash}(\mathsf{hp}, \mathsf{lpar}, \chi, w)) > \epsilon \cdot \nu] = \mathrm{negl}(n)\ ,$$

where the probability is taken over the choice of $\mathsf{hk} \leftarrow \mathsf{HashKG}(\mathsf{lpar})$ and the random coins of Hash *and* $\mathsf{ProjHash}$*.*[11]

[11] Contrary to previously known SPHFs, some of our SPHFs have randomized algorithms Hash and $\mathsf{ProjHash}$.

– ***Smoothness.*** *For any positive integer n, if* $(\mathsf{ltrap}, \mathsf{lpar}) \leftarrow \mathsf{Setup.lpar}(1^n)$*, with overwhelming probability over the randomness of* Setup.lpar*, for all $\chi \in \mathcal{X} \setminus \mathcal{L}_{\mathsf{lpar}}$ the following distributions have statistical distance negligible in n:*

$$\{(\mathsf{lpar}, \chi, \mathsf{hp}, \mathsf{H}) \mid \mathsf{hk} \leftarrow \mathsf{HashKG}(\mathsf{lpar}),\ \mathsf{H} \leftarrow \mathsf{Hash}(\mathsf{hk}, \mathsf{lpar}, \chi),\ \mathsf{hp} = \mathsf{ProjKG}(\mathsf{hk}, \mathsf{lpar}, \chi)\} \ ,$$
$$\{(\mathsf{lpar}, \chi, \mathsf{hp}, \mathsf{H}) \mid \mathsf{hk} \leftarrow \mathsf{HashKG}(\mathsf{lpar}),\ \mathsf{H} \leftarrow \{0,1\}^{\nu},\ \mathsf{hp} = \mathsf{ProjKG}(\mathsf{hk}, \mathsf{lpar}, \chi)\} \ .$$

Finally, an approximate SPHF is called an SPHF if it is 0-correct. In that case, we also say that the SPHF is *statistically correct*.

Approximate KV-SPHFs. For some applications, in particular the one-round PAKE from [KV11], a stronger notion of SPHF is required, where the projection key hp does not depend on the word χ and the smoothness holds even if the word is chosen adaptively after seeing the projection key. Following the terminology of [BBC+13b], we call such (approximate) SPHFs, (approximate) KV-SPHF.[12] We formally define approximate KV-SPHFs in Appendix B.1.

Approximate universal bit-PHFs and bit-KV-PHFs. Instead of directly building (approximate) (KV-)SPHF, we actually build what we call (approximate) universal bit-(KV-)PHF.

Definition 2.14. *An approximate universal bit projective hash function (bit-PHF) is defined as in Definition 2.13 except that the hash values are bits ($\nu = 1$), and that approximate correctness and smoothness are replaced by the following properties:*

– ***Approximate correctness.*** *The bit-PHF is ϵ-correct if for any positive integer n, if* $(\mathsf{ltrap}, \mathsf{lpar}) \leftarrow \mathsf{Setup.lpar}(1^n)$*, with overwhelming probability over the randomness of* Setup.lpar*, for any $\chi \in \bar{\mathcal{L}}_{\mathsf{lpar},\mathsf{ltrap}}$:*

$$\Pr_{\mathsf{hk}}\left[\mathsf{Hash}(\mathsf{hk}, \mathsf{lpar}, \chi) = \mathsf{ProjHash}(\mathsf{hp}, \mathsf{lpar}, \chi, w)\right] \geq 1 - \epsilon \ ,$$

where the probability is taken over the choice of $\mathsf{hk} \leftarrow \mathsf{HashKG}(\mathsf{lpar})$ *and the random coins of* Hash *and* ProjHash.

– ***Universality.*** *The bit-PHF is ϵ-universal*[13] *if, for any positive integer n, if* $(\mathsf{ltrap}, \mathsf{lpar}) \leftarrow \mathsf{Setup.lpar}(1^n)$*, with overwhelming probability over the randomness of* Setup.lpar*, for any word $\chi \in \mathcal{X} \setminus \mathcal{L}_{\mathsf{lpar}}$, any projection key* hp*:*

$$\left|2 \cdot \Pr_{\mathsf{hk}}\left[\mathsf{Hash}(\mathsf{hk}, \mathsf{lpar}, \chi) = 1 \mid \mathsf{hp} = \mathsf{ProjKG}(\mathsf{hk}, \mathsf{lpar}, \chi)\right] - 1\right| \leq \epsilon \ ,$$

where the probability is taken over the choice of $\mathsf{hk} \leftarrow \mathsf{HashKG}(\mathsf{lpar})$ *and the random coins of* Hash*. The bit-SPHF is said to be* statistically universal *if it is* negl(n)*-universal. Otherwise, the bit-SPHF is said to be* imperfectly universal*.*

An approximate bit-PHF is called a bit-PHF if it is negl(n)-correct. In that case, the bit-PHF is said to be *statistically correct*. Furthermore, an (approximate) bit-PHF is called an (approximate) bit-KV-PHF, if hp does not depend on the word χ.

[12] The letters KV in the name KV-SPHF correspond to the initials of the authors of [KV11]. SPHFs defined in [KV09] are not KV-SPHF.

[13] Our definition of universality is equivalent to the one of Cramer and Shoup in [CS02], up to the use of language parameters.

From Bit-PHFs to SPHFs. In Appendix B.2, we show how to generically convert an approximate ϵ-correct $\mathrm{negl}(n)$-universal bit-PHF into an approximate $(\epsilon+\epsilon')$-correct SPHF (for any positive constant ϵ') and then into an SPHF. This is used in our first construction in Section 3. These transformations were implicit in [KV09]. We should point out that even if the original bit-PHF was a bit-KV-PHF, the resulting (approximate) SPHF would still not be a KV-SPHF: its projection key depends on the word χ. If there was way to avoid this restriction, we actually would get the first one-round key exchange based on LWE with polynomial modulus.

In Appendix B.2, we also show how to generically convert an ϵ-universal bit-KV-PHF into a KV-SPHF, by amplifying the smoothness or universality property (assuming $1-\epsilon \geq 1/\operatorname{poly}(n)$). We should point out that the original bit-KV-SPHF is supposed to be statistically correct, contrary to the previous construction where it could only be approximately correct.

We recall that the above transformations were summarized in Fig. 1 together with our results.

3 SPHF for IND-CCA2 LWE Ciphertexts

As we have shown in Section 2.3, there exists a generic transformation from approximate bit-SPHF to a regular approximate SPHF or even classical SPHF. So, in this section, we are going to focus on building such an approximate bit-SPHF. For the sake of simplicity, in this section we often call such an approximate bit-PHF simply a bit-PHF.

3.1 Languages and Natural Bit-PHF

Languages. We want to construct an (approximate) bit-PHF for the language of ciphertexts (Example 2.12) for our IND-CCA2 LWE encryption à la Micciancio-Peikert described in Section 2.2. More generally our approach works with typical trapdoored LWE encryption schemes [GPV08, CHKP10].

We first remark that it is sufficient to construct a bit-PHF for the tag-IND-CCA2 version, i.e., for the following languages:

$$\begin{aligned}\bar{\mathscr{L}} &= \{(u,\boldsymbol{c},\mu) \mid \exists \boldsymbol{s},\boldsymbol{e},\ \boldsymbol{c} \leftarrow \mathsf{Encrypt}(\boldsymbol{A}_0,u,\mu;\boldsymbol{s},\boldsymbol{e})\} \subseteq \{(u,\boldsymbol{c},\mu) \mid d(\boldsymbol{c}-\mathsf{Encode}(\mu),\ \Lambda(\boldsymbol{A}_u)) \leq B\}\ ,\\ \mathscr{L} &= \{(u,\boldsymbol{c},\mu) \mid \mathsf{Decrypt}(\boldsymbol{T},u,\boldsymbol{c}) = \mu\} \qquad\qquad = \{(u,\boldsymbol{c},\mu) \mid d(\boldsymbol{c}-\mathsf{Encode}(\mu),\ \Lambda(\boldsymbol{A}_u)) \leq B'\}\ ,\end{aligned}$$

where $u \in \mathcal{U}$, $\boldsymbol{c} \in \mathbb{Z}_q^m$, $\mu \in \{0,1\}$, $(\mathsf{Itrap},\mathsf{Ipar}) = (\boldsymbol{T},\boldsymbol{A}_0) \leftarrow \mathsf{TrapGen}(1^n,1^m,q) = \mathsf{Setup.Ipar}(1^n)$, and where $\mathsf{Encrypt}$, $\mathsf{Decrypt}$, B, and B' are defined in Section 2.2. Indeed, the signature parts, used to transform the tag-IND-CCA2 encryption scheme into a labeled IND-CCA2 encryption scheme (see Appendix A.2), can be publicly checked by anyone, therefore one can generically adapt the bit-PHF by overriding Hash to a fresh uniform random value when the signature is invalid.

We can now fix the tag $u \in \mathcal{U}$ for the rest of this section, and will simply denote $\boldsymbol{A}$ for $\boldsymbol{A}_u$ and Λ for $\Lambda(\boldsymbol{A}_u)$. Also, note that $(u,\boldsymbol{c},1) \in \bar{\mathscr{L}}$ (resp. $\mathscr{L}$) is equivalent to $(u,\boldsymbol{c}-\mathsf{Encode}(1),0) \in \bar{\mathscr{L}}$ (resp $\mathscr{L}$). Therefore we can focus only on the languages of ciphertexts of 0 for a fixed tag u: we restrict our languages to

$$\begin{aligned}\bar{\mathscr{L}} &= \{\boldsymbol{c} \in \mathbb{Z}_q^m \mid \exists \boldsymbol{s},\boldsymbol{e},\ \boldsymbol{c} \leftarrow \mathsf{Encrypt}(\boldsymbol{A_0},0,u;\boldsymbol{s},\boldsymbol{e})\} \subseteq \{\boldsymbol{c} \in \mathbb{Z}_q^m \mid d(\boldsymbol{c},\Lambda) \leq B\}\ ,\\ \mathscr{L} &= \{\boldsymbol{c} \in \mathbb{Z}_q^m \mid \mathsf{Decrypt}(\boldsymbol{T},\boldsymbol{c},u) = 0\} \qquad\qquad = \{\boldsymbol{c} \in \mathbb{Z}_q^m \mid d(\boldsymbol{c},\Lambda) \leq B'\}\ ,\end{aligned}$$

for the rest of this section.

Natural Bit-PHF. A natural approach to define an approximate bit-PHF is the following:

- $\mathsf{HashKG}(\boldsymbol{A})$ outputs $\mathsf{hk} = \boldsymbol{h} \leftarrow D_{\mathbb{Z},s}^m$;
- $\mathsf{ProjKG}(\boldsymbol{h},\boldsymbol{A})$ outputs $\mathsf{hp} = \boldsymbol{p} = \boldsymbol{A}^t\boldsymbol{h}$;
- $\mathsf{Hash}(\boldsymbol{h},\boldsymbol{A},\boldsymbol{c})$ outputs $\mathsf{H} = R(\langle \boldsymbol{h},\boldsymbol{c}\rangle)$;
- $\mathsf{ProjHash}(\boldsymbol{p},\boldsymbol{A},\boldsymbol{c},(\boldsymbol{s},\boldsymbol{e}))$ outputs $\mathsf{pH} = R(\langle \boldsymbol{p},\boldsymbol{s}\rangle)$;

where R is a *rounding* function to be chosen later and $s > 0$ is a parameter to be chosen later too.

3.2 Universality

Naive approach. For now let us just assume $R : \mathbb{Z}_q \to \mathbb{Z}_2$ to be the usual rounding function $R(x) = \lfloor 2x/q \rceil \bmod 2$, as in [KV09]. If the protocol was ran honestly, note that:

$$\langle \boldsymbol{h}, \boldsymbol{c} \rangle = \boldsymbol{h}^t(\boldsymbol{A}\boldsymbol{s} + \boldsymbol{e}) = \langle \boldsymbol{p}, \boldsymbol{s} \rangle + \langle \boldsymbol{h}, \boldsymbol{e} \rangle \approx \langle \boldsymbol{p}, \boldsymbol{s} \rangle \ ,$$

which guarantees correctness whenever $c \in \bar{\mathscr{L}}$. Indeed $\langle \boldsymbol{h}, \boldsymbol{c} \rangle$ is almost uniform for large enough parameter s, therefore $R(\langle \boldsymbol{h}, \boldsymbol{c} \rangle) = R(\langle \boldsymbol{p}, \boldsymbol{s} \rangle)$ will hold except with probability $\approx 2|\langle \boldsymbol{h}, \boldsymbol{e} \rangle|/q$.

For universality, we need to prove that $\mathsf{Hash}(\boldsymbol{h}, \boldsymbol{A}, \boldsymbol{c}) = \langle \boldsymbol{h}, \boldsymbol{c} \rangle$ is uniform given the knowledge of $\boldsymbol{A}, \boldsymbol{p}$ and $\boldsymbol{c}$, when $\boldsymbol{c} \notin \mathscr{L}$. Unfortunately, this seems to require a stronger assumption than $\boldsymbol{c} \notin \mathscr{L}$, more precisely, that $j \cdot \boldsymbol{c} \notin \mathscr{L}$ for all $j \in \mathbb{Z}_q$: this is the key lemma in [KV09] (from [GPV08]).

Lemma 3.1 ([GPV08, Lemma 5.3], [KV09, Lemma 2]). *Let $s \geq \sqrt{q} \cdot \omega(\sqrt{\log n})$. Then, for most matrices $\boldsymbol{A} \in \mathbb{Z}_q^{m \times n}$ the following is true: if $\boldsymbol{c} \in \mathbb{Z}_q^m$ is such that for all non-zero $j \in \mathbb{Z}_q$, $d(j\boldsymbol{c}, \Lambda(\boldsymbol{A})) \geq \sqrt{q}/4$, then the smoothing parameter $\eta_\epsilon(\Lambda^\perp([\boldsymbol{A}|\boldsymbol{c}]))$ is below s for some negligible function $\epsilon = \mathrm{negl}(n)$.*

In particular, for $h \leftarrow D_{\mathbb{Z},s}^m$ the distribution $(\boldsymbol{h}^t\boldsymbol{A}, \boldsymbol{h}^t\boldsymbol{c})$ is negligibly close to uniform over $\mathbb{Z}_q^{n+1}$.

The caveat is that it is necessary not only for $\boldsymbol{c}$ to be far from Λ, but also for all its non-zero multiples modulo q: the language is extended to $\mathscr{L}' = \{\boldsymbol{c} \mid \exists j \in \mathbb{Z}_q, j\boldsymbol{c} \in \mathscr{L}\}$. Algorithmically, the price to pay is that the decryption function must be changed, and that the usual LWE decryption now must be attempted for each multiple $j\boldsymbol{c}$ of $\boldsymbol{c}$ to ensure universality for words outside $\mathscr{L}'$. This makes the new decryption very inefficient since q is typically quite a large $\mathrm{poly}(n)$. This change of language is also a technical hassle for constructing protocols above the bit-PHF (or the resulting SPHF).

Note that the above lemma ensures uniformity of $\langle \boldsymbol{h}, \boldsymbol{c} \rangle$, while we only need the uniformity of $R(\langle \boldsymbol{h}, \boldsymbol{c} \rangle)$. Naturally, one may wonder whether the condition that $j\boldsymbol{c}$ is far from Λ for all $j \neq 0$ is truly necessary or whether it is an artifact of the proof. To answer this question, let us first explore two case studies.

Two case studies. Let us take a look at the special case where q is even, and where $\boldsymbol{c}$ is a perfect encryption of 1: $\boldsymbol{c} = \boldsymbol{A}\boldsymbol{s} + (0, \ldots, 0, q/2)^t$ (so that $\boldsymbol{c} \notin \mathscr{L}$ with overwhelming probability over the choice of the public key ek). We then observe that

$$\langle \boldsymbol{h}, \boldsymbol{c} \rangle = \langle \boldsymbol{p}, \boldsymbol{s} \rangle + (h_m \bmod 2) \cdot q/2 \ ,$$

where h_m is the last coordinate of h. In particular, the distribution of $\langle \boldsymbol{h}, \boldsymbol{c} \rangle$, when $\boldsymbol{h} \leftarrow D_{\mathbb{Z},s}^m$, is concentrated on merely 2 values out of q and is therefore far from uniform.

Yet, assuming s is twice as large as the smoothing parameter of $\mathbb{Z}$, we note that h_m is uniform modulo 2. In that case we observe that while $\langle \boldsymbol{h}, \boldsymbol{c} \rangle$ is not itself uniform, the rounding $R(\langle \boldsymbol{h}, \boldsymbol{c} \rangle)$ is uniform when choosing the typical rounding function $R : x \in \mathbb{Z}_q \mapsto \lfloor 2x/q \rceil \mod 2$, regardless of the value of $\langle \boldsymbol{p}, \boldsymbol{s} \rangle$. So it seems that the rounding function does not only help in turning approximate correctness into exact correctness, but it can also improve universality of the scheme as well!

Unfortunately, we can not always expect statistical universality from this trick. Now assume that q is divisible by 3, and set $\boldsymbol{c} = \boldsymbol{A}\boldsymbol{s} + (0, \ldots, 0, q/3)^t$ (again, $\boldsymbol{c} \notin \mathscr{L}$ with overwhelming probability over the choice of the public key ek). This time,

$$\langle \boldsymbol{h}, \boldsymbol{c} \rangle = \langle \boldsymbol{p}, \boldsymbol{s} \rangle + (h_m \bmod 3) \cdot q/3$$

is uniformly distributed over three values, separated by $q/3$. In particular $R(\langle \boldsymbol{h}, \boldsymbol{c} \rangle)$ will take one value with probability 1/3, and the other value with probability 2/3. Despite imperfect universality, this still guarantees some entropy in $\mathsf{Hash}(\boldsymbol{h}, \boldsymbol{A}, \boldsymbol{c})$ knowing $\boldsymbol{A}$, $\boldsymbol{c}$, and $\boldsymbol{p}$.

But what should happen in more general cases?

Harmonic analysis. Let us fix $\boldsymbol{p} \in \mathbb{Z}_q^n$ and $\boldsymbol{c} \in \mathbb{Z}_q^m$. For the rest of the section, we restrict the rounding function R to have binary values $\{0,1\}$, yet this function may be probabilistic.

We want to study the conditional probability $P = \Pr[R(\langle \boldsymbol{h}, \boldsymbol{c}\rangle) = 1 \mid \boldsymbol{h}^t\boldsymbol{A} = \boldsymbol{p}^t]$, where the probability is taken over the randomness of R and the distribution of $\boldsymbol{h}$ (conditioned on $\boldsymbol{h}^t\boldsymbol{A} = \boldsymbol{p}^t$); we want P to be not too far from $1/2$ when $\boldsymbol{c} \notin \mathscr{L}$. For $x \in \mathbb{Z}$, denote by $r(x)$ the probability that $R(x \bmod q) = 1$. Because $r : \mathbb{Z} \to [0,1]$ is q-periodic, it can be interpolated over the reals by a function of the form:

$$r = \sum_{j \in \mathbb{Z}_q} \hat{r}_j \cdot e_{j/q} \ ,$$

where the complex values $\hat{r}_j \in \mathbb{C}$ are the Fourier coefficients of $r\colon \mathbb{Z} \to [0,1]$. Note that as we are only interested in the restriction of r on $\mathbb{Z}$ (which is q-periodic), we only need q harmonics to fully describe r. Also note that $r(x) \in [0,1]$ for all $x \in \mathbb{Z}_q$, so that $|\hat{r}_j| \le 1$ for all j.

We rewrite:

$$P = \sum_{\boldsymbol{h} \in \Lambda_{\boldsymbol{p}}^\perp} \frac{\rho_s(\boldsymbol{h})}{\rho_s(\Lambda_{\boldsymbol{p}}^\perp)} \cdot r(\langle \boldsymbol{h}, \boldsymbol{c}\rangle) = \frac{1}{\rho_s(\Lambda_{\boldsymbol{p}}^\perp)} \sum_{j \in \mathbb{Z}_q} \hat{r}_j \sum_{\boldsymbol{h} \in \Lambda^\perp} (\rho_s \cdot e_{j\boldsymbol{c}/q})(\boldsymbol{h} + \boldsymbol{h}_0) \ ,$$

where $\boldsymbol{h}_0$ is any vector of the coset $\Lambda_{\boldsymbol{p}}^\perp$. We will now apply the Poisson Summation Formula (Lemma 2.6): $f(\Lambda^\perp) = \det((\Lambda^\perp)^*)\hat{f}((\Lambda^\perp)^*) = \det(\frac{1}{q}\Lambda)\hat{f}(\frac{1}{q}\Lambda)$. Set $f(\boldsymbol{h}) = (\rho_s \cdot e_{j\boldsymbol{c}/q})(\boldsymbol{h} + \boldsymbol{h}_0)$. We have:

$$\hat{f} = \widehat{\rho_s \cdot e_{\boldsymbol{v}}} \cdot e_{\boldsymbol{h}_0} = s^m \rho_{1/s,\boldsymbol{v}} \cdot e_{\boldsymbol{h}_0} \ .$$

We proceed:

$$P = \frac{\det((\Lambda^\perp)^*)s^m}{\rho_s(\Lambda_{\boldsymbol{p}}^\perp)} \sum_{j \in \mathbb{Z}_q} \hat{r}_j \cdot (\rho_{1/s,j\boldsymbol{c}/q} \cdot e_{\boldsymbol{h}_0})\left(\frac{1}{q}\Lambda\right) = \frac{\det((\Lambda^\perp)^*)s^m}{\rho_s(\Lambda_{\boldsymbol{p}}^\perp)} \sum_{j \in \mathbb{Z}_q} \hat{r}_j \cdot \sum_{\boldsymbol{y} \in \Lambda} (\rho_{q/s,j\boldsymbol{c}} \cdot e_{\boldsymbol{h}_0/q})(\boldsymbol{y}) \, .$$

Assuming $s \ge \eta_\epsilon(\Lambda^\perp)$ for some negligible ϵ ensures that $\frac{\det((\Lambda^\perp)^*)s^m}{\rho_s(\Lambda_{\boldsymbol{p}}^\perp)} = 1 + O(\epsilon)$ by Lemma 2.3. We shall split the sum into three parts:

- $j = 0$, $\boldsymbol{y} = \boldsymbol{0}$, contributing exactly $\hat{r}_0$ (where $\hat{r}_0 = \frac{1}{q}\sum_{x \in \mathbb{Z}_q} r(x) \in [0,1]$),
- $j = 0$, $\boldsymbol{y} \ne \boldsymbol{0}$, contributing at most $|\hat{r}_0|\rho_{q/s}(\Lambda \setminus \{\boldsymbol{0}\})$ in absolute value,
- $j \ne 0$, $\boldsymbol{y} \ne \boldsymbol{0}$, contributing at most $|\hat{r}_j|\rho_{q/s}(\Lambda - j\boldsymbol{c})$ in absolute value for each j.

We can now bound P:

$$\left|\frac{P}{1 - O(\epsilon)} - \hat{r}_0\right| \le |\hat{r}_0|\rho_{q/s}(\Lambda \setminus \{\boldsymbol{0}\}) + \sum_{j \in \mathbb{Z}_q \setminus \{0\}} |\hat{r}_j|\rho_{q/s}(\Lambda - j\boldsymbol{c}) \ .$$

We now want to bound the right-hand side using Lemma 2.1, with $c = 1$ for simplicity. Fix $j \in \mathbb{Z}_q \setminus \{0\}$, and let $\alpha = q\sqrt{m}/s$. If $\alpha < d(j\boldsymbol{c}, \Lambda)$, then $(\Lambda - j\boldsymbol{c}) \setminus \alpha\mathcal{B} = (\Lambda - j\boldsymbol{c})$. Also, note that $\rho_{q/s}(\Lambda) = \rho_{1/s}(\frac{1}{q}\Lambda) = \rho_{1/s}((\Lambda^\perp)^*)$. So, as long as $s \ge \eta_\epsilon(\Lambda^\perp)$ for some negligible ϵ (which we already assumed earlier), it holds that $\rho_{q/s}(\Lambda) \le 1 + \epsilon$ by definition of $\eta_\epsilon(\Lambda^\perp)$. Under those conditions, $\rho_{q/s}(\Lambda - j\boldsymbol{c}) = \rho_{q/s}((\Lambda - j\boldsymbol{c}) \setminus \alpha\mathcal{B}) \le 2C^m\rho_{q/s}(\Lambda) \le 2C^m(1 + \epsilon)$ is negligible. Using Lemma 2.1, we deduce the following:

Theorem 3.2. *Fix $\boldsymbol{A} \in \mathbb{Z}_q^{m \times n}$, $\boldsymbol{c} \in \mathbb{Z}_q^m$, and $\boldsymbol{p} \in \mathbb{Z}_q^n$, where m is polynomial in n. Fix a probabilistic rounding function $R : \mathbb{Z}_q \to \{0,1\}$ such that for all $x \in \mathbb{Z}_q$,*

$$\Pr[R(x) = 1] = r(x) = \sum_{j \in J} \hat{r}_j e_{j/q}(x) \ ,$$

where $J \subseteq \mathbb{Z}_q$ and $\hat{r}_j \in \mathbb{C}$. Let $s \ge \eta_\epsilon(\Lambda^\perp(\boldsymbol{A}))$ for some $\epsilon = \mathrm{negl}(n)$. Assume furthermore that

$$\forall j \in J \setminus \{0\},\ s \cdot d(j\boldsymbol{c}, \Lambda(\boldsymbol{A})) > q\sqrt{m} \ .$$

Denote $P(\boldsymbol{c}) = \Pr[R(\langle \boldsymbol{h}, \boldsymbol{c}\rangle) = 1 \mid \boldsymbol{h}^t\boldsymbol{A} = \boldsymbol{p}^t]$, where the probability is taken over the randomness of R, and the distribution of $\boldsymbol{h} \leftarrow D_{\mathbb{Z},s}^m$, conditioned on $\boldsymbol{h}^t\boldsymbol{A} = \boldsymbol{p}^t$. Then :

$$|P(\boldsymbol{c}) - \hat{r}_0| \le (2 + O(\epsilon))\,|J|\,C^m + O(\epsilon) \quad \textit{where} \quad C = \sqrt{2\pi e} \cdot e^{-\pi} < 1 \ .$$

Setting up the rounding function. If one wishes to avoid having to attempt decryption of many multiples of the ciphertext $\boldsymbol{c}$, one should choose a probabilistic rounding function with a small number of harmonics.

In particular, the typical deterministic rounding function $R(x) = \lfloor 2x/q \rceil \bmod 2$ —the so-called square-signal— and has harmonic coefficients $\hat{r}_j$ decreasing as $\Theta(1/j)$ in absolute value. With such a rounding function, one would still need to attempt trapdoor inversion for $q/2$ many multiples of $\boldsymbol{c}$, as it was already the case in [KV09].

On the contrary, one may easily avoid costly harmonics by setting the rounding function so that $2r(x) = 1 + \cos(2\pi x/q)$, which has Fourier coefficients $\hat{r}_0 = 1/2$, $\hat{r}_1 = \hat{r}_{-1} = 1/4$, and $\hat{r}_j = 0$ for any other j.[14]

In order to prove universality, assume $\boldsymbol{c} \notin \mathscr{L}$, so that $d(\boldsymbol{c}, \Lambda) \geq B'$ by definition. Therefore, whenever $\alpha = q\sqrt{m}/s < B'$, we have $(\Lambda - \boldsymbol{c}) \setminus (\alpha\mathcal{B}) = (\Lambda - \boldsymbol{c})$.

Corollary 3.3. *Let $\boldsymbol{A} \in \mathbb{Z}_q^{m\times n}$ with $m = \Theta(n \log q)$, and fix $\boldsymbol{p} \in \mathbb{Z}_q^n$. Let $B' = q/\Theta(\sqrt{m})$, and $\mathscr{L} = \{\boldsymbol{c} \in \mathbb{Z}_q^m \mid d(\boldsymbol{c}, \Lambda(\boldsymbol{A})) \leq B'\}$. Suppose that R satisfies:*

$$\Pr[R(x) = 1] = r(x) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi x}{q}\right) \ ,$$

and let $s \geq \eta_\epsilon(\Lambda^\perp(\boldsymbol{A}))$ for some $\epsilon = \mathrm{negl}(n)$. Suppose also that: $s > \frac{q\sqrt{m}}{B'}$.

Denote again $P(\boldsymbol{c}) = \Pr[R(\langle \boldsymbol{h}, \boldsymbol{c}\rangle) = 1 \mid \boldsymbol{h}^t\boldsymbol{A} = \boldsymbol{p}^t]$, where the probability is taken over the randomness of R, and the distribution of $\boldsymbol{h} \leftarrow D_{\mathbb{Z},s}^m$, conditioned on $\boldsymbol{h}^t\boldsymbol{A} = \boldsymbol{p}^t$. Then, for all $\boldsymbol{c} \notin \mathscr{L}$:

$$|2P(\boldsymbol{c}) - 1| \leq 2\,(6 + O(\epsilon))\,C^m + O(\epsilon) \leq \mathrm{negl}(n) \ ,$$

where $C = \sqrt{2\pi e} \cdot e^{-\pi} < 1$.

3.3 Approximate Correctness

Let us check that the scheme above achieves approximate correctness, that is, for all $\boldsymbol{c} \in \bar{\mathscr{L}}$, $\mathsf{Hash}(\boldsymbol{h}, \boldsymbol{A}, \boldsymbol{c}) = \mathsf{ProjHash}(\boldsymbol{p}, \boldsymbol{A}, \boldsymbol{c}, (\boldsymbol{s}, \boldsymbol{e}))$ with probability substantially greater than $1/2$. Using our rounding function R, this means that we want $R(\langle \boldsymbol{h}, \boldsymbol{c}\rangle)$ and $R(\langle \boldsymbol{p}, \boldsymbol{s}\rangle)$ to output the same bit with some probability Q substantially greater than $1/2$, where the two applications of R use independent coins.

Recall that $r(x)$ is the probability that the rounding function R outputs 1 on input x, and that for $\boldsymbol{c} \in \bar{\mathscr{L}}$, we can write $\langle \boldsymbol{h}, \boldsymbol{c}\rangle = \langle \boldsymbol{p}, \boldsymbol{s}\rangle + \langle \boldsymbol{h}, \boldsymbol{e}\rangle$, where $\boldsymbol{c} = \boldsymbol{A}\boldsymbol{s} + \boldsymbol{e}$. We argue that as long as $\langle \boldsymbol{h}, \boldsymbol{e}\rangle$ is small with respect to q, then our scheme achieves approximate correctness:

Lemma 3.4. *Fix $\boldsymbol{A} \in \mathbb{Z}_q^{m\times n}$ and $\boldsymbol{c} = \boldsymbol{A}\boldsymbol{s} + \boldsymbol{e} \in \bar{\mathscr{L}}$, where m and q are polynomial in n, and where $\|\boldsymbol{e}\| \leq B = 2t\sqrt{m}$. Let $s \geq \eta_\epsilon(\Lambda^\perp(\boldsymbol{A}))$ for some $\epsilon = \mathrm{negl}(n)$. Assume that R satisfies:*

$$\Pr[R(x) = 1] = r(x) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi x}{q}\right) \ , \tag{1}$$

Let Q be the probability that $R(\langle \boldsymbol{h}, \boldsymbol{c}\rangle; \mathsf{coins}_1)$ and $R(\langle \boldsymbol{A}^t\boldsymbol{h}, \boldsymbol{s}\rangle; \mathsf{coins}_2)$ output the same bit, over the randomness of $\boldsymbol{h} \leftarrow D_{\mathbb{Z},s}^m$, and the randomness of the two independent coins coins_1 and coins_2 used by R. Suppose furthermore that:

$$tsm = o(q) \ .$$

Then:

$$Q = \frac{3}{4} + o(1) \ .$$

[14] Of course, one could also obtain perfect universality by setting a constant rounding function $r(x) = 1/2$, and even avoid the first harmonic, but there is no way to reach correctness even with amplification in that case.

Proof. As $s \geq \eta_\epsilon(\Lambda^\perp)$ for $\epsilon = \mathrm{negl}(n)$, the distribution of $\boldsymbol{h}^t\boldsymbol{A}$, when $\boldsymbol{h} \leftarrow D^m_{\mathbb{Z},s}$, is at negligible statistical distance from uniform.

Therefore, Q is negligibly close to $\Pr[R(x;\mathsf{coins}_1) = R(x + \langle \boldsymbol{h}, \boldsymbol{e}\rangle; \mathsf{coins}_2)]$ where the probability is taken over uniform $x \in \mathbb{Z}_q$, $\boldsymbol{h} \leftarrow D^m_{\mathbb{Z},s}$, and the randomness of the two independent coins coins_1 and coins_2 used by R.

Then:

$$\begin{aligned} Q &= \frac{1}{q}\sum_{x\in\mathbb{Z}_q} (r(x)r(x+\langle \boldsymbol{h},\boldsymbol{e}\rangle) + (1-r(x))(1-r(x+\langle \boldsymbol{h},\boldsymbol{e}\rangle))) + \mathrm{negl}(n) \\ &= \frac{1}{2} + \frac{1}{q}\sum_{x\in\mathbb{Z}_q} \frac{1}{2}\cos\left(2\pi\frac{x}{q}\right)\cos\left(2\pi\frac{x+\langle \boldsymbol{h},\boldsymbol{e}\rangle}{q}\right) + \mathrm{negl}(n) \ . \end{aligned}$$

As $tsm = o(q)$, we have $\langle \boldsymbol{h}, \boldsymbol{e}\rangle = o(q)$ with overwhelming probability. As cos is a Lipschitz continuous function, we can approximate the sum by an integral:

$$Q = \frac{1}{2} + \frac{1}{2}\int_0^1 \cos^2(2\pi x)dx + o(1) = \frac{3}{4} + o(1) \ .$$

This concludes the proof. □

3.4 Wrap-up

Consider the bit-PHF described in Section 3.1 instantiating R with the cosine rounding function (Eq. (1)), together with the encryption scheme of Section 2.2. Let us now show that all the parameters can be instantiated to satisfy security and correctness of the encryption scheme, simultaneously with statistical universality and approximate correctness of the bit-PHF.

IND-CCA2. To base the security of the scheme described in Section 2.2 on $\mathrm{LWE}_{\chi,q}$ for $\chi = D_{\mathbb{Z},\sigma}$ and $\sigma = 2\sqrt{n}$,[15] we apply Theorem 2.9 with

$$m = \Theta(n \log q) \ , \quad t = \sqrt{mn} \cdot \omega(\sqrt{\log n}) \ .$$

Decryption Correctness. For the encryption scheme to be correct, we want $B < B'$, recalling that $B \coloneqq 2t\sqrt{m}$ and $B' \coloneqq q/\Theta(\sqrt{m})$.

Universality. In Corollary 3.3, we used the hypothesis $s \geq \eta_\epsilon(\Lambda^\perp(\boldsymbol{A}_u))$ for some negligible ϵ. Assuming $s \geq \Theta(\sqrt{n})$, one can apply Lemma 2.11, to ensure the above hypothesis for $\epsilon = 2^{-n}$ simultaneously for all $u \in \mathcal{U}$ except with probability 2^{-n} over the randomness of $\mathsf{TrapGen}$.

Still in Corollary 3.3, we also needed $s > q\sqrt{m}/B'$, where $B' = q/\Theta(\sqrt{m})$. This holds for $s = \Theta(m)$.

Approximate correctness. For Lemma 3.4, we assumed that $tsm = o(q)$. Equivalently, it is sufficient that $sm^{3/2}n^{1/2}\omega(\sqrt{\log n}) = o(q)$.

Summary. Therefore, all the desired conditions can be satisfied with:

$$q = \tilde{\Theta}(n^3) \ , \quad m = \tilde{\Theta}(n) \ , \quad s = \tilde{\Theta}(n) \ , \quad t = \tilde{\Theta}(n) \ .$$

We have proved the following:

[15] This is the smallest parameter σ for which $\mathrm{LWE}_{\chi,q}$ is known reduce to a worst-case problem. One may of course choose to use a different width for the LWE error, and derive different appropriate parameters.

Theorem 3.5. *Set parameters $q = \tilde{\Theta}(n^3), m = \tilde{\Theta}(n), s = \tilde{\Theta}(n), t = \tilde{\Theta}(n)$. Define a probabilistic rounding function $R : \mathbb{Z}_q \to \{0,1\}$ such that:*

$$\Pr[R(x) = 1] = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi x}{q}\right) \ .$$

Then,

- *the encryption scheme of Section 2.2 is correct and tag-IND-CCA2 under the hardness of $LWE_{\chi,q}$ for $\chi = D_{\mathbb{Z},2\sqrt{n}}$;*
- *the bit-PHF described in Section 3.1 achieves statistical universality and $(1/4 - o(1))$-correctness.*

4 KV-SPHF for IND-CPA LWE Ciphertexts

4.1 Overview

In the previous section, we built a bit-PHF with $\mathrm{negl}(n)$-universality but approximate correctness. Even though the correctness can be amplified (as described in Appendix B.2), the transformation inherently makes the new projection key depend on the word we want to hash, even if that was not the case for the initial bit-PHF.

We now build a bit-PHF with statistical correctness and K-universality for some universal constant $K < 1$ (but using a super-polynomial LWE modulus q). The main benefit of such a construction is that amplifying universality can be done regardless of the word we want to hash, that is, the projection key will not depend on the word. When the tag u of the ciphertext $\boldsymbol{c}$ is known in advance or is constant (in which case, the encryption scheme is only IND-CPA instead of IND-CCA2), we therefore get a bit-KV-PHF which can be transformed into a KV-SPHF. This is the first KV-SPHF for any lattice-based language.

We use the same natural approach as described in Section 3.1. The only differences with the construction in the previous section are the probabilistic rounding function we use, and the parameters necessary to argue correctness and universality. Recall that in the last section, we used a rounding function with only low order harmonics to get $\mathrm{negl}(n)$-universality.

The starting point is the observation that, for the naive square rounding introduced in the previous section, the correctness is statistical, but clearly not $\mathrm{negl}(n)$-universal, depending on which word $\boldsymbol{c}$ is hashed (as seen in the examples in Section 3.2, where $j \cdot \boldsymbol{c}$ is close to Λ for some $j \in \mathbb{Z}_q \setminus \{0\}$). However, the distribution of $R(\langle \boldsymbol{h}, \boldsymbol{c}\rangle)$ conditioned on $\boldsymbol{h}^t\boldsymbol{A}$ might still have enough entropy to give us K-universality, for some constant $K < 1$. In other words, we can hope that $|2 \cdot \Pr[R(\langle \boldsymbol{h}, \boldsymbol{c}\rangle) = 1 \mid \boldsymbol{p}] - 1| \leq K$ for all $\boldsymbol{c} \in \mathbb{Z}_q^m$.

Let $R^\sharp$ be a rounding function defined by: $R^\sharp(x) = 1 + \lfloor 2x/q \rceil \bmod 2$, that is:

$$\forall x \in [-q/2, q/2], \quad R^\sharp(x) = \begin{cases} 1 & \text{if } |x| \in [-q/4, q/4] \\ 0 & \text{otherwise.} \end{cases} ,$$

Using this rounding function gives good correctness: when $s \geq \eta_\epsilon(\Lambda^\perp)$, $\langle \boldsymbol{h}, \boldsymbol{c}\rangle$ is statistically close to uniform in $[-q/2, q/2]$, and therefore $R^\sharp(\langle \boldsymbol{h}, \boldsymbol{c}\rangle)$ is a uniform bit up to some statistical distance $O(\epsilon + 1/q)$ (due to the fact that q is odd). So for super-polynomial q, we get *statistical correctness* using $R^\sharp$ as rounding function, as long as $\langle \boldsymbol{h}, \boldsymbol{e}\rangle$ is sufficiently small with respect to q.

For *universality*, we express the probability distribution defined by $R^\sharp$, seen as a q-periodic function over $\mathbb{R}$, as a Fourier series:

$$\forall x \in [-q/2, q/2], \quad r^\sharp(x) \coloneqq \Pr[R^\sharp(x) = 1] = \sum_{j \in \mathbb{Z}} \hat{r}_j^\sharp \cdot e_{j/q}(x) \ ,$$

where $\hat{r}_j^\sharp$ are the Fourier coefficients of the q-periodic function $r^\sharp \colon \mathbb{R} \to \mathbb{R}$.

However, one can show that $|\hat{r}_j^\sharp| = \Theta(1/j)$ (for odd integers j).. Therefore, it is not clear how to show universality with a similar analysis as in Section 3.2: the total contribution of harmonics j such that $j \cdot \boldsymbol{c}$ is close to Λ could potentially be arbitrarily large!

To solve this issue, we consider a new rounding function R, which has the same probability distribution as $R^\sharp$ but on a negligible fraction of points (so that statistical correctness is preserved), and such that its Fourier coefficients of high enough order have small enough amplitude.

Then, we use the observation that the set of integers j such that $j \cdot \boldsymbol{c}$ is in Λ is an ideal of $\mathbb{Z}$, which is proper if $\boldsymbol{c}$ itself is not in Λ. More generally, the set of *small* integers $j \in \mathbb{Z}$ such that $j \cdot \boldsymbol{c}$ is *close* to Λ is contained in an ideal of $\mathbb{Z}$; furthermore, if $\boldsymbol{c}$ is far from Λ, then this ideal is a proper ideal of $\mathbb{Z}$. This will allow us to discard all harmonics whose order is not in this ideal. As we will show, the remaining harmonics necessarily have destructive interferences, which allows us to establish K-universality for some constant $K < 1$.

The roadmap follows. First, in Section 4.2, we smooth the discontinuities of the probability distribution of the square rounding function $r^\sharp$ so that the Fourier coefficients of high order have small magnitude, but such that we keep statistical correctness. Then to prove universality, in Section 4.3, we show that for $\boldsymbol{c}$ far from Λ, the set of small $j \in \mathbb{Z}$ such that $j \cdot \boldsymbol{c}$ is close to Λ is contained in a proper ideal of $\mathbb{Z}$. Finally, in Section 4.4 we show that the distribution of $R(\langle \boldsymbol{h}, \boldsymbol{c} \rangle)$ conditioned on $\boldsymbol{h}^t \boldsymbol{A}$ has some bounded min entropy.

4.2 Smoothing the Discontinuities: a New Rounding Function

In the following, unless specified otherwise, we will see $\mathbb{Z}_q$ as embedded in $\{\lceil -q/2 \rceil, \dots, \lfloor q/2 \rfloor\}$, and the canonical period we use for q-periodic functions will be $[-q/2, q/2]$. Recall that $r^\sharp$ satisfies:

$$\forall x \in [-q/2, q/2], \quad r^\sharp(x) = \begin{cases} 1 & \text{if } |x| \in [-q/4, q/4] \\ 0 & \text{otherwise.} \end{cases} ,$$

In particular, $r^\sharp$ has two discontinuities on $q/4$ and on $-q/4$. To smooth those discontinuities, we consider the convolution product of the square signal $r^\sharp$ with a rectangular signal of appropriate width T such that $T/q = \mathrm{negl}(n)$. More precisely, consider the q-periodic function $r^\flat$ defined on $[-q/2, q/2]$ by:

$$\forall x \in [-q/2, q/2], \quad r^\flat(x) = \begin{cases} \frac{1}{2T} & \text{if } |x| \leq T \\ 0 & \text{otherwise.} \end{cases} ,$$

We define a new rounding function R such that for all $x \in \mathbb{R}$ (see Fig. 2):

$$\Pr[R(x) = 1] \coloneqq r(x) \coloneqq (r^\sharp \odot r^\flat)(x) \coloneqq \int_{-q/2}^{q/2} r^\sharp(u) \cdot r^\flat(x-u)\, du ,$$

where, in this context, $\odot$ corresponds to the convolution of q-periodic functions.

Intuitively, this corresponds to replace the discontinuities on $r^\sharp(\pm q/4)$ by a linear slope ranging from $\pm q/4 - T$ to $\pm q/4 + T$ (see Fig. 2). Therefore, over $[-q/2, q/2]$, the functions r and $r^\sharp$ only differ on at most $4\lceil T \rceil$ integer points (the points on the slope). Recall that if $s \geq \eta_\epsilon(\Lambda^\perp)$, then $\langle \boldsymbol{h}, \boldsymbol{c} \rangle$ is statistically close to uniform in $\{\lceil -q/2 \rceil, \dots, \lfloor q/2 \rfloor\}$. Therefore, if $\langle \boldsymbol{h}, \boldsymbol{e} \rangle / q$ and T/q are negligible, then:

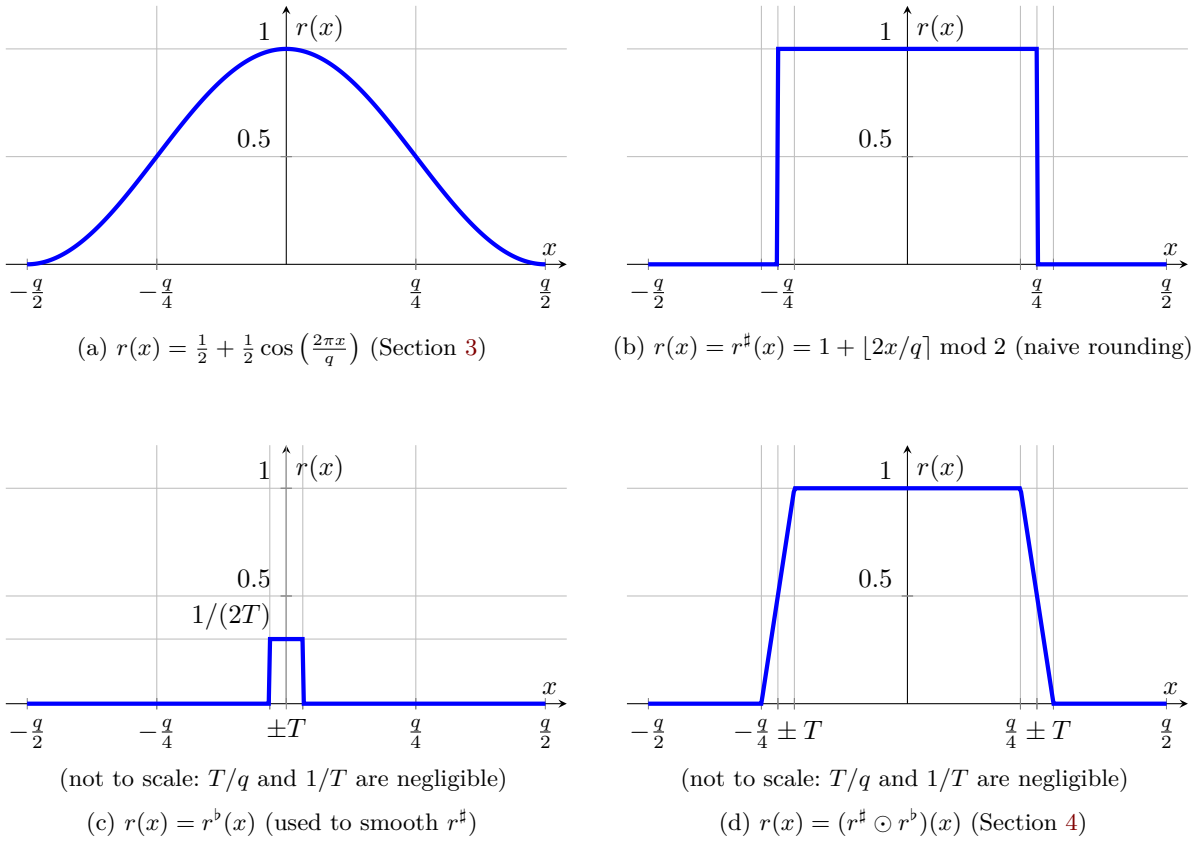
$$\Pr[R(\langle \boldsymbol{h}, \boldsymbol{c} \rangle) \neq R(\langle \boldsymbol{p}, \boldsymbol{s} \rangle)] \leq \mathrm{negl}(n) ,$$

and we get statistical correctness using such a rounding function.

Lemma 4.1 (Correctness). *Suppose that $s \geq \eta_\epsilon(\Lambda^\perp)$, $tsm/q = \mathrm{negl}(n)$, and $T/q = \mathrm{negl}(n)$. Assume that R satisfies:*

$$\Pr[R(x) = 1] = (r^\sharp \odot r^\flat)(x) .$$

Then the approximate bit-PHF defined in Section 3.1 achieves statistical correctness.



(a) $r(x) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi x}{q}\right)$ (Section 3)

(b) $r(x) = r^\sharp(x) = 1 + \lfloor 2x/q \rceil \bmod 2$ (naive rounding)

(not to scale: T/q and $1/T$ are negligible)

(c) $r(x) = r^\flat(x)$ (used to smooth $r^\sharp$)

(not to scale: T/q and $1/T$ are negligible)

(d) $r(x) = (r^\sharp \odot r^\flat)(x)$ (Section 4)

Fig. 2: Probability that the rounding functions $R(x)$ of Sections 3 and 4 output 1

Also, r is q-periodic, and can therefore be expressed as a Fourier series:

$$\forall x \in [-q/2, q/2], r(x) = \sum_{j\in\mathbb{Z}} \hat{r}_j e_{j/q}(x) \ ,$$

with Fourier coefficients $\hat{r}_j$. As $r = r^\sharp \odot r^\flat$, we have $\hat{r}_j = q \cdot \hat{r}^\sharp_j \cdot \hat{r}^\flat_j$ for $j \in \mathbb{Z}$, where $\hat{r}^\sharp_j$ and $\hat{r}^\flat_j$ are the Fourier coefficients of the q-periodic functions $r^\sharp$ and $r^\flat$ respectively. Thus, $\hat{r}_0 = 1/2$, and for $j \in \mathbb{Z} \setminus \{0\}$, the jth harmonic of r is:

$$\hat{r}_j = \frac{q}{2\pi^2 T j^2} \cdot \sin(\pi j/2) \cdot \sin(2\pi T j/q) \leq \frac{q}{19 T j^2} \ . \tag{2}$$

4.3 Inclusion of Contributing Harmonics in a Proper Ideal

In the following, we focus on showing that even though we do not have $\mathrm{negl}(n)$-universality using this new rounding function, we still have some K-universality for some constant $K < 1$ (that we can amplify).

We start by a simple useful lemma:

Lemma 4.2. *Let $N = kq/T$ for some k. Then $\sum_{j\in\mathbb{Z},\ |j|>N} |\hat{r}_j| \leq 1/k$.*

Proof. It follows from Eq. (2) and the fact that for all $N > 2$: $\sum_{k=N}^{+\infty} \frac{1}{k^2} \leq \sum_{k=N}^{+\infty} \left(\frac{1}{k-1} - \frac{1}{k}\right) = \frac{1}{N-1}$. □

Suppose now that $d(\boldsymbol{c}, \Lambda) \geq B'$. Consider the set of $j \in \mathbb{Z}$ such that $d(j \cdot \boldsymbol{c}, \Lambda) \leq \delta$ for some appropriately chosen δ. Let $P = P(\boldsymbol{c}) = \Pr[R(\langle \boldsymbol{h}, \boldsymbol{c}\rangle) = 1 \mid \boldsymbol{h}^t \boldsymbol{A} = \boldsymbol{p}^t]$, for our new rounding function R. For any $\boldsymbol{h_0} \in \Lambda^\perp_{\boldsymbol{p}}$, we can show similarly to Section 3.2, that:

$$P = \frac{\det((\Lambda^\perp)^*) s^m}{\rho_s(\Lambda^\perp_{\boldsymbol{p}})} \sum_{j\in\mathbb{Z}} \hat{r}_j \sum_{y\in\Lambda} (\rho_{q/s, jc} \cdot e_{\boldsymbol{h_0}/q})(\boldsymbol{y}) \ , \tag{3}$$

where $\frac{\det((\Lambda^\perp)^*)s^m}{\rho_s(\Lambda^\perp_{\boldsymbol{p}})} = (1+O(\epsilon))$ as long as $s \geq \eta_\epsilon(\Lambda^\perp)$. Note that $\sum_{|j|\geq N} |\hat{r}_j|$ can be made arbitrarily small for appropriate N, by Lemma 4.2. Thus only the terms of the sum corresponding to $|j| \leq N$ will have a substantial contribution to the sum above (recall that $\rho_{q/s}(\Lambda - j\boldsymbol{c}) \leq 1+\epsilon$ for all $\boldsymbol{c}$, for appropriate parameters). Therefore we only consider those small j such that $|j| < N$ for some appropriately chosen N (with respect to q). Furthermore, for large enough δ, the terms corresponding to indices j such that $d(j\cdot\boldsymbol{c}, \Lambda) \geq \delta$ also have a negligible contribution to the sum by Lemma 2.1. For appropriate parameters N and δ to be instantiated later, let:

$$J = \{j \in \mathbb{Z} \mid |j| < N \wedge d(j\cdot\boldsymbol{c}) \leq \delta\} \ . \tag{4}$$

As a subset of $\mathbb{Z}$, J is contained in the ideal $j_0\mathbb{Z}$ of $\mathbb{Z}$, where $j_0 = \gcd(J)$. Let us show that it is a proper ideal of $\mathbb{Z}$, i.e. $j_0 \neq 1$. To do so, we rely on the existence of small Bézout coefficients.

Lemma 4.3 (Corollary of [MH94, Theorem 9]). *Let $a_0, \ldots, a_k \in \mathbb{Z}$, and let $g = \gcd(a_0, \ldots, a_k)$. Then there exists $u_0, \ldots, u_k \in \mathbb{Z}$ such that the following conditions hold:*

$$\sum_{i=0}^{k} u_i a_i = g \ , \qquad \sum_{i=0}^{k} |u_i| \leq \frac{k}{2} \max |a_i| \ .$$

We can now prove that J is a proper ideal of $\mathbb{Z}$:

Lemma 4.4. *Suppose that $\delta N^2 < B'$. Then, for $\boldsymbol{c} \in \mathbb{Z}_q^m$ such that $d(\boldsymbol{c}, \Lambda) > B'$, the set $J = \{j < N \mid d(j\cdot\boldsymbol{c}, \Lambda) \leq \delta\}$ is contained in a proper ideal of $\mathbb{Z}$.*

Proof. Let $j_0 = \gcd(J)$. By definition, $J \subseteq j_0\mathbb{Z}$. Suppose by contradiction that $j_0 = 1$. By Lemma 4.3, there exists a set of integers $\{u_j, j \in J\}$ such that $\sum_{j\in J} u_j \cdot j = 1$ and then $\sum_{j\in J} u_j \cdot (j\cdot\boldsymbol{c}) = \boldsymbol{c}$. But by definition of J, $d(j\cdot\boldsymbol{c}, \Lambda) \leq \delta$ for all $j \in J$, and therefore:

$$d(\boldsymbol{c}, \Lambda) \leq \delta \cdot \sum_{j\in J} |u_j| \leq \frac{\delta \# J}{2} \max_{j\in J} |j| \leq \delta N^2 < B' \ ,$$

which is absurd as we assumed $d(\boldsymbol{c}, \Lambda) > B'$. □

4.4 Imperfect Universality from Destructive Interferences

We now want to quantify how biased $R(\langle \boldsymbol{h}, \boldsymbol{c}\rangle)$ conditioned on $\boldsymbol{h}^t\boldsymbol{A}$ can be when $\boldsymbol{c}$ is far from Λ. We start from Eq. (3):

$$P = \frac{\det((\Lambda^\perp)^*)s^m}{\rho_s(\Lambda^\perp_{\boldsymbol{p}})} \sum_{j\in\mathbb{Z}} \hat{r}_j \sum_{y\in\Lambda} (\rho_{q/s, j\boldsymbol{c}} \cdot e_{\boldsymbol{h}_0/q})(\boldsymbol{y}) \ ,$$

where $\frac{\det((\Lambda^\perp)^*)s^m}{\rho_s(\Lambda^\perp_{\boldsymbol{p}})} = 1 + O(\epsilon)$ as long as $s \geq \eta_\epsilon(\Lambda^\perp)$.

We split the sum into three parts $P = P_1 + P_2 + P_3$:

- P_1. $|j| > N \wedge j \notin j_0\mathbb{Z}$: those indices have a negligible contribution to the sum by Lemma 4.2.
- P_2. $|j| \leq N \wedge j \notin j_0\mathbb{Z}$: those indices contribute negligibly since $\rho_{q/s}(\Lambda - j\boldsymbol{c})$ is small as $j\boldsymbol{c}$ is far from Λ (by definition of δ and $J \subset j_0\mathbb{Z}$).
- P_3. $j \in j_0\mathbb{Z}$: the contributing terms. Unlike the previous ones we won't use absolute bounds for each term, and must consider destructive interferences.

It remains to study

$$P_3 = \sum_{\boldsymbol{h}\in\Lambda^\perp_{\boldsymbol{p}}} \frac{\rho_s(\boldsymbol{h})}{\rho_s(\Lambda^\perp_{\boldsymbol{p}})} \sum_{j\in j_0\mathbb{Z}} \hat{r}_j e_{j/q}(\langle \boldsymbol{h}, \boldsymbol{c}\rangle) \ .$$

If we were to have $j_0 = 1$ (i.e. $j_0\mathbb{Z} = \mathbb{Z}$), we could compute the inner sum simply by inverse Fourier transform, evaluating r at $x = \langle \boldsymbol{h}, \boldsymbol{c} \rangle$. Instead, we note that selecting only the harmonics in $j_0\mathbb{Z}$, corresponds in the temporal domain to averaging the function r over all its temporal shifts by multiples of q/j_0. More formally, recall the identity:

$$\sum_{k=0}^{j_0-1} e_{j/j_0}(k) = \begin{cases} j_0 & \text{if } j \in j_0\mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

We may now rewrite:

$$\sum_{j \in j_0\mathbb{Z}} \widehat{r}_j e_{j/q}(x) = \frac{1}{j_0} \sum_{j \in \mathbb{Z}} \widehat{r}_j e_{j/q}(x) \sum_{k=0}^{j_0-1} e_{j/j_0}(k) = \frac{1}{j_0} \sum_{k=0}^{j_0-1} r(x + k\frac{q}{j_0}) \ ,$$

Note that $\frac{1}{j_0}\sum_{k=0}^{j_0-1} r^\sharp(x + k\frac{q}{j_0})$ is not too far away from $1/2$: if j_0 is even, this is exactly $1/2$, and if $j_0 = 2k+1$, this is either k/j_0 or $(k+1)/j_0$, which is at distance $1/(2j_0)$ from $1/2$. Overall, the distance to $1/2$ is therefore always less than $1/6$ as $j_0 \neq 1$ by Lemma 4.4.

Furthermore, this conclusion also holds for $\frac{1}{j_0}\sum_{k=0}^{j_0-1} r(x + k\frac{q}{j_0})$, as we have, by construction:

$$\forall x \in [-q/2, q/2], |r(x) - 1/2| \leq |r^\sharp(x) - 1/2| \ .$$

Therefore, P_3 is also not too far from $1/2$ as a convex combination of values not too far from $1/2$, so that we would have imperfect universality. More precisely:

$$|P_3 - 1/2| \leq 1/6 \ .$$

Putting everything together, we can quantify the distance from P to $1/2$:

Theorem 4.5 (Universality). *Let $\boldsymbol{A} \in \mathbb{Z}_q^{m \times n}$ with $m = \Theta(n \log q)$, and fix $\boldsymbol{p} \in \mathbb{Z}_q^n$. Let $B' = q/\Theta(\sqrt{m})$, and $\mathscr{L} = \{\boldsymbol{c} \in \mathbb{Z}_q^m \mid d(\boldsymbol{c}, \Lambda(\boldsymbol{A})) \leq B'\}$. Let R be as defined in Section 4.2 and let $s \geq \eta_\epsilon(\Lambda^\perp(\boldsymbol{A}))$ for some $\epsilon = \mathrm{negl}(n)$. Suppose also that parameters T, N, δ, and k satisfy $\delta > \frac{q\sqrt{m}}{s}$, $N = \frac{kq}{T}$, and $\delta N^2 < B'$.*

Denote again $P(\boldsymbol{c}) = \Pr[R(\langle \boldsymbol{h}, \boldsymbol{c} \rangle) = 1 \mid \boldsymbol{h}^t\boldsymbol{A} = \boldsymbol{p}^t]$, where the probability is taken over the randomness of R, and the distribution of $\boldsymbol{h} \leftarrow D_{\mathbb{Z},s}^m$, conditioned on $\boldsymbol{h}^t\boldsymbol{A} = \boldsymbol{p}^t$. Then, for all $\boldsymbol{c} \notin \mathscr{L}$:

$$|P(\boldsymbol{c}) - 1/2| \leq \frac{1}{6} + (1 + O(\epsilon)) \left(\frac{1}{k} + 4NC^m \right) \ ,$$

where $C = \sqrt{2\pi e} \cdot e^{-\pi} < 1$.

Remark 4.6. Informally, this theorem states that the second case study of Section 3.2 is essentially the worst case.

Proof. Writing $P = P_1 + P_2 + P_3$ as above, we showed that $|P_3 - 1/2| \leq 1/6$. Moreover, as $s \geq \eta_\epsilon(\Lambda^\perp(\boldsymbol{A}))$, we have:

$$\frac{\det((\Lambda^\perp)^*) s^m}{\rho_s(\Lambda_{\boldsymbol{p}}^\perp)} = 1 + O(\epsilon) \ ,$$

and, for any $j \in \mathbb{Z}$ and $\boldsymbol{c}$, we also have:

$$|\sum_{\boldsymbol{y} \in \Lambda} (\rho_{q/s, j\boldsymbol{c}} \cdot e_{\boldsymbol{h}_0/q})(\boldsymbol{y})| \leq \rho_{q/s}(\Lambda - j\boldsymbol{c}) \leq 1 + \epsilon \ .$$

Therefore, by Lemma 4.2, and as $\epsilon = \mathrm{negl}(n)$, we have:

$$|P_1| \leq (1 + O(\epsilon))(1 + \epsilon) \sum_{|j| > N} |\hat{r}_j| \leq \frac{1 + O(\epsilon)}{k} \ .$$

Furthermore, as $\delta > \frac{q\sqrt{m}}{s}$, and $|\hat{r}_j| \leq 1$ for all j, Lemma 2.1 gives us that:

$$|P_2| \leq 4NC^m(1 + O(\epsilon)) \ ,$$

which concludes the proof. ∎

4.5 Wrap-up

Let us now show that all the parameters can be instantiated to get approximate smoothness and correctness for the SPHF, using a rounding function R defined by $\Pr[R(x) = 1] = r^\sharp \odot r^\flat(x)$.

IND-CPA. To apply Theorem 2.9 with Remark 2.10, we can use:

$$m = \Theta(n \log q), \quad t = \sqrt{mn} \cdot \omega(\sqrt{\log n}) \ .$$

Decryption Correctness. For the encryption scheme to be correct, we want $B < B'$, with $B = 2t\sqrt{m}$ and $B' = q/\Theta(\sqrt{m})$.

Correctness. For correctness of the bit-PHF, we need a super-polynomial modulus q, and require T/q to be negligible. Furthermore, we need tsm/q to be negligible, so that $\langle \boldsymbol{h}, \boldsymbol{e} \rangle$ can only take a negligible fraction of values in $\mathbb{Z}_q$. Also, we need $s \geq \eta_\epsilon(\Lambda^\perp(\boldsymbol{A}_u))$, which is satisfied with high probability by Lemma 2.11 for $\epsilon = 2^{-n}$ as long as $s \geq \Theta(\sqrt{n})$.

Bounding the amplitude of high frequencies. The parameter N which upper bounds the elements of J must be taken so that $\sum_{|j| \geq N} |\hat{r}_j|$ is small. By Lemma 4.2, by taking $N = kq/T$, this sum is $\leq 1/k$.

Threshold distance to Λ defining J. The parameter δ, which denotes how close $j \cdot \boldsymbol{c}$ is close to Λ for $j \in J$ (Eq. (4)) has to be chosen so that $N \cdot \rho_{q/s}(\Lambda - \boldsymbol{v})$ must be small whenever $d(\boldsymbol{v}, \Lambda) \geq \delta$. As in the analysis for the cosine rounding function, setting $\delta = q\sqrt{m}/s$ implies that $\rho_{q/s}(\Lambda - \boldsymbol{v}) \leq 2C^m(1 + O(\epsilon))$ by Lemma 2.1.

Showing that $j_0 \neq 1$. We also required $\delta N^2 < B'$ to conclude that J was included in a proper ideal of $\mathbb{Z}$. As we have $\delta N^2 = \Theta\left(\frac{q^3 k\sqrt{m}}{sT^2}\right)$, this holds as long as $s \geq \Omega(\frac{mk^2q^2}{T^2})$.

Putting everything together, we get the following theorem:

Theorem 4.7. *Suppose $q = O(2^n)$ to be superpolynomial in n, $m = \Theta(n \log q)$. Set parameters:*

- *T such that T/q and q/T^2 are both negligible in n (using $T = q^{2/3}$ for instance),*
- *$k = \Theta(n)$,*
- *$s \geq \Theta(\sqrt{n})$ such that $s/q = \mathrm{negl}(n)$ and $s = \Omega(\frac{mk^2q^2}{T^2})$, which exists by construction of T.*

Define a probabilistic rounding function $R : \mathbb{Z}_q \to \{0, 1\}$ such that:

$$\Pr[R(x) = 1] = r^\sharp \odot r^\flat(x) \ .$$

Then the bit-PHF described in Section 3.1 achieves $(1/3 + o(1))$-universality and statistical correctness.

Proof. The theorem follows from the discussion above and Theorem 4.5 using:

- $N = kq/T$ such that NC^m is negligible in n (which exists as long as $q = O(2^n)$),
- $\delta = \frac{q\sqrt{m}}{s}$.

□

5 Applications

In this section, we present several applications of our new construction. It underlines the importance of revisiting this primitive.

5.1 Password-Authenticated Key Exchange

Gennaro and Lindell proposed in [GL06] a generic framework for building PAKE protocols based on SPHF and IND-CCA2 encryption scheme. Later in [KV09], Katz and Vaikuntanathan refined it to be compatible with approximate SPHF over a CCA2-secure encryption scheme.

We briefly recall it in Fig. 3. We assume a common reference string is established before any executions of the protocol take place. The common reference string consists of a public key for a CCA2-secure encryption scheme that has an associated ϵ-correct approximate SPHF (i.e., an ϵ-correct approximate SPHF for the language defined in Example 2.12). No party in the system is assumed to know the secret key associated with it.

Common reference string: A common reference string $\mathsf{lpar} = \mathsf{ek}$ corresponding to a public key of the IND-CCA2 public key encryption.

Common private input: A password π

Messages:

1. Party P_i chooses a key-pair $(\mathsf{VK}, \mathsf{SK})$ for a strongly unforgeable one-time signature scheme, sets $\mathsf{label}_i = \mathsf{VK} \| P_i \| P_j \| 1$ and $\mathsf{label}_j = \mathsf{VK} \| P_i \| P_j \| 2$, computes $C_i = \mathsf{Encrypt}(\mathsf{lpar}, \mathsf{label}_i, \pi; w_i)$ and sends (VK, C_i) to P_j;
2. Party P_j receives (VK, C_i), sets $\mathsf{label}_i = \mathsf{VK} \| P_i \| P_j \| 1$ and $\mathsf{label}_j = \mathsf{VK} \| P_i \| P_j \| 2$, checks that C_i is of the proper format, and does the following:
   (a) Computes $\mathsf{hk}_j \leftarrow \mathsf{HashKG}(\mathsf{lpar})$ and $\mathsf{hp}_j \leftarrow \mathsf{ProjKG}(\mathsf{hk}_j, \mathsf{lpar}, (\mathsf{label}_i, C_i, \pi))$,
   (b) Generates an encryption $C_j = \mathsf{Encrypt}(\mathsf{lpar}, \mathsf{label}_j, \pi; w_j)$.
   P_j then sends (hp_j, C_j) to P_i.
3. Party P_i receives (hp_j, C_j), checks that C_j is of the proper format and does the following:
   (a) Computes $\mathsf{hk}_i \leftarrow \mathsf{HashKG}(\mathsf{lpar})$ and $\mathsf{hp}_i \leftarrow \mathsf{ProjKG}(\mathsf{hk}_i, \mathsf{lpar}, (\mathsf{label}_j, C_j, \pi))$,
   (b) Picks $\mathsf{sk}_i \leftarrow \{0,1\}^\ell$, sets $c = \mathsf{ECC}(\mathsf{sk}_i)$,
   (c) Computes $\Delta = \mathsf{Hash}(\mathsf{hk}_i, \mathsf{lpar}, (\mathsf{label}_j, C_j, \pi)) \oplus \mathsf{ProjHash}(\mathsf{hp}_j, \mathsf{lpar}, (\mathsf{label}_i, C_i, \pi), w_i) \oplus c$,
   (d) Computes a signature σ of $(C_i, C_j, \mathsf{hp}_i, \mathsf{hp}_j, \Delta)$ under SK.
   P_i then sends (Δ, σ) to P_j.

Session Key Definition:

- P_i possesses the session key sk_i.
- P_j checks the validity of the one time signature σ, and computes the session key $\mathsf{sk}_j = \mathsf{ECC}^{-1}(\mathsf{ProjHash}(\mathsf{hp}_i, \mathsf{lpar}, (\mathsf{label}_j, C_j, \pi), w_j) \oplus \mathsf{Hash}(\mathsf{hk}_j, \mathsf{lpar}_j, (\mathsf{label}_i, C_i, \pi)) \oplus \Delta)$.

Fig. 3: Generic PAKE from an approximate SPHF $(\mathsf{HashKG}, \mathsf{ProjKG}, \mathsf{Hash}, \mathsf{ProjHash})$ for an IND-CCA2 encryption scheme $(\mathsf{KeyGen}, \mathsf{Encrypt}, \mathsf{Decrypt})$

Assuming that the SPHF is ϵ-correct, and that ECC is an error correcting code with a $2 - \epsilon$ correction rate, the resulting construction achieves correctness. If in addition the projection keys of the SPHF do not depend on the message $M = \pi$ (as it is the case with our construction), as shown in [KV09], the security of the protocol is directly inherited from the smoothness of the SPHF and the IND-CCA2 security of the encryption scheme. Indeed for a passive adversary the session key is pseudo-random, and an active adversary can not efficiently construct a new ciphertext decrypting to the client password, nor guessing the session key for something that does not decrypt to the said password.

Using our new construction. We can instantiate the construction using the encryption scheme à la Micciancio-Peikert in Section 2.2 together with an approximate SPHF generically derived (via the transformation in Appendix B.2) from the approximate bit-SPHF constructed in Section 3. This allow us to achieve a PAKE protocol in three flows, with a polynomial modulus.

Moving to a 2-round PAKE An interesting optimization in cryptography is to reduce the number of rounds, so that each user only has to speak once. Is it possible to achieve a PAKE, where each user sends simply one flow?

In [ABP15b], the authors improved the Gennaro-Lindell framework to reduce its number of rounds to two. Their construction (called GL-PAKE) requires an IND-CPA encryption with a KV-SPHF on one hand, and an IND-PCA (Indistinguishable against Plaintext-Checkable Attacks) encryption with a regular SPHF on the other hand. As for the 2-round PAKE above, the projection key of the SPHF is supposed to be independent of the message $M = \pi$. Every IND-CCA2 encryption being also IND-PCA, we can meet the requirements. The scheme is described in Figure 4.

Due to the nature of the SPHF over lattices, we have different languages for correctness and smoothness, however the proof can straightforwardly be adapted to handle this particularity.

Common reference string: A common reference string ($\mathsf{lpar} = \mathsf{ek}$, $\mathsf{lpar}' = \mathsf{ek}'$), where ek is a public key of the IND-CCA2 encryption scheme $(\mathsf{KeyGen}, \mathsf{Encrypt}, \mathsf{Decrypt})$ and ek' is a public key of the IND-CPA encryption scheme $(\mathsf{KeyGen}', \mathsf{Encrypt}', \mathsf{Decrypt}')$.

Common private input: A password π

Messages:

1. Party P_i does the following:
   (a) Computes $\mathsf{hk}_j \leftarrow \mathsf{HashKG}'(\mathsf{lpar}')$ and $\mathsf{hp}_j \leftarrow \mathsf{ProjKG}'(\mathsf{hk}_j, \mathsf{lpar}')$,
   (b) Sets $\mathsf{label} = P_i \| P_j \| \mathsf{hp}_i$ and generates $C_i = \mathsf{Encrypt}(\mathsf{lpar}, \mathsf{label}, \pi; w_i)$.
   P_i then sends (hp_i, C_i) to P_j.
2. Party P_i receives (hp_i, C_i), checks that C_i is of the proper format and does the following:
   (a) Generates $C_j = \mathsf{Encrypt}'(\mathsf{lpar}', \perp, \pi; w_j)$,
   (b) Sets $\mathsf{label} = P_i \| P_j \| \mathsf{hp}_i$ and compute $\mathsf{hk}_i \leftarrow \mathsf{HashKG}(\mathsf{lpar})$ and $\mathsf{hp}_j \leftarrow \mathsf{ProjKG}(\mathsf{hk}_j, \mathsf{lpar}, (\mathsf{label}, C_i, \pi))$.
   P_j then sends (hp_j, C_j) to P_i.

Session Key Definition:

- P_i receives (hp_j, C_j), checks that C_j is of the proper format, and computes the session key $\mathsf{sk}_i = \mathsf{ProjHash}(\mathsf{hp}_j, \mathsf{lpar}, (\mathsf{label}, C_i, \pi), w_i) \oplus \mathsf{Hash}'(\mathsf{hk}_i, \mathsf{lpar}', (\perp, C_j, \pi))$.
- P_i computes the session key $\mathsf{sk}_j = \mathsf{Hash}(\mathsf{hk}_i, \mathsf{lpar}, (\mathsf{label}, C_i, \pi)) \oplus \mathsf{ProjHash}'(\mathsf{hp}_j, \mathsf{lpar}', (\perp, C_j, \pi), w_j)$.

Fig. 4: Two-Round PAKE from an IND-CCA2 encryption scheme $(\mathsf{KeyGen}, \mathsf{Encrypt}, \mathsf{Decrypt})$ with an SPHF $(\mathsf{HashKG}, \mathsf{ProjKG}, \mathsf{Hash}, \mathsf{ProjHash})$ (from Section 3) and an IND-CPA encryption scheme $(\mathsf{KeyGen}', \mathsf{Encrypt}', \mathsf{Decrypt}')$ with a KV-SPHF $(\mathsf{HashKG}', \mathsf{ProjKG}', \mathsf{Hash}', \mathsf{ProjHash}')$ (from Section 4)

5.2 Honest-Verifier Zero-Knowledge

It has already been shown in [BP13], that SPHF could be used to produce Honest-Verifier Zero Knowledge proofs. Our construction is compatible with such a technique for all NP languages of the form $\ddot{\mathscr{L}} = \{\ddot{\chi} \mid \exists \ddot{w}, \ddot{\mathscr{R}}(\ddot{\chi}, \ddot{w})\}$ where $\ddot{\mathscr{R}}$ is a polynomial-size circuit. The use of double dots on top of the language and its words is used to distinguish it from the language of the underlying SPHFs.

Generic construction. At a very high level, the prover will simply do a CPA-secure encryption[16] of each wire of the circuit, and then show the correct evaluation at each gate, using SPHFs.

For the sake of simplicity, we suppose that all gates of the circuit $\ddot{\mathscr{R}}$ are NAND gates. We define the following languages $\bar{\mathscr{L}} \subseteq \mathscr{L}$ of ciphertexts C_1, C_2, C_3 encrypting values (b_1, b_2, b_3) so that $b_3 = \mathrm{NAND}(b_1, b_2)$:

$$\bar{\mathscr{L}} = \left\{ (C_1, C_2, C_3) \;\middle|\; \begin{array}{l} \exists \rho_1, \rho_2, \rho_3, b_1, b_2, b_3,\ \forall i \in \{1,2,3\},\ C = \mathsf{Encrypt}(\mathsf{ek}, b_i) \\ \text{and } b_3 = \mathrm{NAND}(b_1, b_2) \end{array} \right\},$$

$$\mathscr{L} = \left\{ (C_1, C_2, C_3) \;\middle|\; \begin{array}{l} \exists b_1, b_2, b_3,\ \forall i \in \{1,2,3\},\ \mathsf{Decrypt}(\mathsf{dk}, C_i) = b_i \\ \text{and } b_3 = \mathrm{NAND}(b_1, b_2) \end{array} \right\},$$

[16] We actually will use our CCA2-secure encryption scheme à la Micciancio-Peikert.

where $(\mathsf{ltrap}, \mathsf{lpar}) = (\mathsf{dk}, \mathsf{ek})$ is a key pair for the CPA-secure encryption scheme. Labels are omitted as they are not used. We suppose that we have an SPHF for the languages $\bar{\mathscr{L}} \subseteq \mathscr{L}$. This SPHF is used to check that the prover encrypted wire values which corresponds to a valid evaluation of a gate. Following the methodology from [BP13, BCPW15], our participants are going to interact as described in Fig. 5.

Common reference string: Encryption key ek for a CPA-secure encryption scheme.
Wire commitments: For each wire i in the circuit $\ddot{\mathscr{R}}$ evaluated on the word $\ddot{\chi}$ for the argument and a witness $\ddot{w}$, the prover is going to do a CPA-secure encryption of its value b in $C_i \leftarrow \mathsf{Encrypt}(\mathsf{ek}, b)$, and keeps the corresponding randomness (or witness) $\rho_i = w_i$. He then sends the corresponding ciphertexts $C = \{C_i\}_i$.
Verifying the gates:

1. For each NAND gate j, linking the wires i_1, i_2 to i_3, the verifier computes $\mathsf{hk}_j, \mathsf{hp}_j, \mathsf{H}_j$ for the SPHF described in the text and the word $\chi = (C_{i_1}, C_{i,2}, C_{i_3})$. The verifier then sends $\mathsf{hp} = \{\mathsf{hp}_j\}_j$.
2. For each hp_j, the prover using w_{i_1}, w_{i_2} and w_{i_3} can now recover pH_j, he then computes $\mathsf{pH} = \bigoplus_j \mathsf{pH}_j$ and sends it to the prover.

Validation:

– The verifier computes $\mathsf{H} = \bigoplus_j \mathsf{H}_j$, and accepts if $\mathsf{H} = \mathsf{pH}$.

Fig. 5: Honest-verifier zero-knowledge argument from SPHFs

Completeness comes directly from the correctness of the underlying SPHFs, while soundness comes from their smoothness. A simulator (for the honest-verifier zero-knowledge property) would encrypt dummy values, computes pH using the various hk_j which under the CPA security of the encryption scheme used would be computationally indistinguishable from the real experiment, ensuring that the previous construction is indeed honest-verifier zero-knowledge.

Instantiation. It remains to show how to constructs SPHFs for the above language $\bar{\mathscr{L}} \subseteq \mathscr{L}$ from our SPHF in Section 3, when the CPA-secure encryption scheme is our CCA2-secure encryption scheme à la Micciancio-Peikert of Section 2.2.

We remark that a set of wire values (b_1, b_2, b_3) corresponds to a valid evaluation of a NAND gate $(b_3 = \mathrm{NAND}(b_1, b_2))$ if and only if $(b_1 = 0 \wedge b_2 = 0 \wedge b_3 = 1) \vee (b_1 = 0 \wedge b_2 = 1 \wedge b_3 = 1) \vee (b_1 = 1 \wedge b_2 = 0 \wedge b_3 = 1) \vee (b_1 = 1 \wedge b_2 = 1 \wedge b_3 = 0)$. Therefore, we can write:

$$\bar{\mathscr{L}} = (\bar{\mathscr{L}}_{1,0} \cap \bar{\mathscr{L}}_{2,0} \cap \bar{\mathscr{L}}_{3,1}) \cup (\bar{\mathscr{L}}_{1,1} \cap \bar{\mathscr{L}}_{2,0} \cap \bar{\mathscr{L}}_{3,0}) \cup (\bar{\mathscr{L}}_{1,0} \cap \bar{\mathscr{L}}_{2,1} \cap \bar{\mathscr{L}}_{3,0}) \cup (\bar{\mathscr{L}}_{1,1} \cap \bar{\mathscr{L}}_{2,1} \cap \bar{\mathscr{L}}_{3,0}),$$
$$\mathscr{L} = (\mathscr{L}_{1,0} \cap \mathscr{L}_{2,0} \cap \mathscr{L}_{3,1}) \cup (\mathscr{L}_{1,1} \cap \mathscr{L}_{2,0} \cap \mathscr{L}_{3,0}) \cup (\mathscr{L}_{1,0} \cap \mathscr{L}_{2,1} \cap \mathscr{L}_{3,0}) \cup (\mathscr{L}_{1,1} \cap \mathscr{L}_{2,1} \cap \mathscr{L}_{3,0}),$$

where:

$$\bar{\mathscr{L}}_{i,b} = \{(C_1, C_2, C_3) \mid \exists \rho,\ C_i = \mathsf{Encrypt}(\mathsf{ek}, b; \rho)\},$$
$$\mathscr{L}_{i,b} = \{(C_1, C_2, C_3) \mid \mathsf{Decrypt}(\mathsf{dk}, C_i) = b\}.$$

We remark that our new SPHF in Section 3 can be easily used to deal with the languages $\bar{\mathscr{L}}_{i,b}$ and $\mathscr{L}_{i,b}$.

It is therefore sufficient to show how to combine SPHFs for the languages $\bar{\mathscr{L}}_{i,b} \subseteq \mathscr{L}_{i,b}$ to get an SPHF for the language $\bar{\mathscr{L}} \subseteq \mathscr{L}$. For that we use the techniques introduced in [ACP09] to handle combinations of SPHFs (for conjunctions "$\cap$" and disjunctions "$\cup$"), and we adapt them to fit our formalism.

Conjunctions and disjunctions of SPHFs. We assume to be given two smooth projective hash functions SPHF_1 and SPHF_2, on the sets corresponding to the languages $\bar{\mathscr{L}}_{\mathsf{lpar}_1}$ and $\bar{\mathscr{L}}_{\mathsf{lpar}_2}$: $\mathrm{SPHF}_i = \{\mathsf{HashKG}_i, \mathsf{ProjKG}_i, \mathsf{Hash}_i, \mathsf{ProjHash}_i\}$.

For a given $\chi \in \mathcal{X}$, we naturally define $\mathsf{hk}_1, \mathsf{hk}_2, \mathsf{hp}_1, \mathsf{hp}_2$ as before.

A smooth projective hash system for the language $\mathcal{L} = \bar{\mathcal{L}}_{\mathsf{lpar}_1} \cap \bar{\mathcal{L}}_{\mathsf{lpar}_2}$ is then defined as follows, if $\chi \in \bar{\mathcal{L}}_{\mathsf{lpar}_1} \cap \bar{\mathcal{L}}_{\mathsf{lpar}_2}$ and w_i is a witness that $\chi \in \bar{\mathcal{L}}_{\mathsf{lpar}_i}$, for both $i = 1, 2$:

- $\mathsf{HashKG}_{\bar{\mathcal{L}}_{\mathsf{lpar}}}(\mathsf{lpar}) = \mathsf{hk} = (\mathsf{hk}_1, \mathsf{hk}_2)$;
- $\mathsf{ProjKG}_{\bar{\mathcal{L}}_{\mathsf{lpar}}}(\mathsf{hk}, \mathsf{lpar}, \chi) = \mathsf{hp} = (\mathsf{hp}_1, \mathsf{hp}_2)$;
- $\mathsf{Hash}_{\bar{\mathcal{L}}_{\mathsf{lpar}}}(\mathsf{hk}, \mathsf{lpar}, \chi) = \mathsf{Hash}_1(\mathsf{hk}_1, \mathsf{lpar}_1, \chi) \oplus \mathsf{Hash}_2(\mathsf{hk}_2, \mathsf{lpar}_2, \chi)$;
- $\mathsf{ProjHash}_{\bar{\mathcal{L}}_{\mathsf{lpar}}}(\mathsf{hp}, \mathsf{lpar}, \chi, (w_1, w_2)) = \mathsf{ProjHash}_1(\mathsf{hp}_1, \mathsf{lpar}_1, \chi, w_1) \oplus \mathsf{ProjHash}_2(\mathsf{hp}_2, \mathsf{lpar}_2, \chi, w_2)$.

The Smoothness is then guaranteed for words outside $\mathcal{L} = \mathcal{L}_{\mathsf{lpar}_1,\mathsf{ltrap}_1} \cap \mathcal{L}_{\mathsf{lpar}_2,\mathsf{ltrap}_2}$.

Similarly, a smooth projective hash system for the language $\mathcal{L} = \bar{\mathcal{L}}_{\mathsf{lpar}_1} \cup \bar{\mathcal{L}}_{\mathsf{lpar}_2}$ is defined as follows, if $\chi \in \bar{\mathcal{L}}_{\mathsf{lpar}_1} \cup \bar{\mathcal{L}}_{\mathsf{lpar}_2}$ and w is a witness that χ belongs to one of the language.

- $\mathsf{HashKG}_{\bar{\mathcal{L}}_{\mathsf{lpar}}}(\mathsf{lpar}) = \mathsf{hk} = (\mathsf{hk}_1, \mathsf{hk}_2)$;
- $\mathsf{ProjKG}_{\bar{\mathcal{L}}_{\mathsf{lpar}}}(\mathsf{hk}, \mathsf{lpar}, \chi) = \mathsf{hp} = (\mathsf{hp}_1, \mathsf{hp}_2, \mathsf{hp}_\Delta)$
  where $\mathsf{hp}_\Delta = \mathsf{Hash}_1(\mathsf{hk}_1, \mathsf{lpar}_1, \chi) \oplus \mathsf{Hash}_2(\mathsf{hk}_2, \mathsf{lpar}_2, \chi))$
- $\mathsf{Hash}_{\bar{\mathcal{L}}_{\mathsf{lpar}}}(\mathsf{hk}, \mathsf{lpar}, \chi) = \mathsf{Hash}_1(\mathsf{hk}_1, \mathsf{lpar}_1, \chi)$;
- $\mathsf{ProjHash}_{\bar{\mathcal{L}}_{\mathsf{lpar}}}(\mathsf{hp}, \mathsf{lpar}, \chi, w) = \begin{cases} \mathsf{ProjHash}_1(\mathsf{hp}_1, \mathsf{lpar}_1, \chi, w) & \text{if } \chi \in \bar{\mathcal{L}}_{\mathsf{lpar}_1} \ , \\ \mathsf{hp}_\Delta \oplus \mathsf{ProjHash}_2(\mathsf{hp}_2, \mathsf{lpar}_2, \chi, w) & \text{if } \chi \in \bar{\mathcal{L}}_{\mathsf{lpar}_2} \ . \end{cases}$

Once again, the Smoothness is then guaranteed for words outside $\mathcal{L} = \mathcal{L}_{\mathsf{lpar}_1,\mathsf{ltrap}_1} \cup \mathcal{L}_{\mathsf{lpar}_2,\mathsf{ltrap}_2}$.

5.3 Witness Encryption

Another application of our previous SPHF would be in the domain of witness encryption [GGSW13] for statements derived from the language of ciphertexts as defined in Example 2.12. [17]

Definition 5.1. *Let $(\bar{\mathcal{L}}_{\mathsf{lpar}} \subseteq \mathcal{L}_{\mathsf{lpar},\mathsf{ltrap}} \subseteq \mathcal{X}_{\mathsf{lpar}})_{\mathsf{lpar},\mathsf{ltrap}}$ be languages defined as before. A witness encryption scheme for these languages is defined by the two probabilistic polynomial-time algorithms: $(\mathsf{Encrypt}_{\mathsf{WE}}, \mathsf{Decrypt}_{\mathsf{WE}})$, where:*

- $\mathsf{Encrypt}_{\mathsf{WE}}(1^n, \chi, M)$ *generates a ciphertext C from a plaintext M, a security parameter n, and a word $\chi \in \mathcal{X}$.*
- $\mathsf{Decrypt}_{\mathsf{WE}}(C, w)$ *decrypts the ciphertext C into M using the witness.*

It has to satisfy the two following properties:

- ***Correctness.*** *For any security parameter n, message $M \in \{0,1\}$, and $\chi \in \bar{\mathcal{L}}_{\mathsf{lpar}}$ such that $\bar{\mathcal{R}}(\chi, w)$ holds, we have*

$$\Pr[\mathsf{Decrypt}_{\mathsf{WE}}(\mathsf{Encrypt}_{\mathsf{WE}}(1^n, \chi, M), w) = M] \geq 1 - \mathrm{negl}(n) \ .$$

- ***Soundness.*** *For any probabilistic polynomial-time adversary $\mathcal{A}$, there exists a negligible function* negl(.) *such that for any positive integer n, if $(\mathsf{ltrap}, \mathsf{lpar}) \leftarrow \mathsf{Setup.lpar}(1^n)$, with overwhelming probability over the randomness of* Setup.lpar*, for any $\chi \notin \mathcal{L}_{\mathsf{lpar},\mathsf{ltrap}}$:*

$$\Pr_{\mathcal{A}}[\mathsf{Encrypt}_{\mathsf{WE}}(1^n, \chi, 0) = 1] - \Pr_{\mathcal{A}}[\mathsf{Encrypt}_{\mathsf{WE}}(1^n, \chi, 1) = 1] < \mathrm{negl}(n) \ .$$

In the original definition [GGSW13], there was a voluntary gap between the soundness and correctness, as nothing is said for words in the language for no known witnesses. Over lattice-based schemes, it is natural to extend the gap, by considering $\bar{\mathcal{L}}_{\mathsf{lpar}}$ for the correctness, while defining the soundness for $\mathcal{L}_{\mathsf{lpar},\mathsf{ltrap}}$, as in our new definition. Another minor difference is the introduction of language parameters $(\mathsf{ltrap}, \mathsf{lpar})$, as we are considering only restricted languages

[17] The concept of using SPHF to generically build Witness Encryption was already mentioned as folklore in the introduction of [ABP15a], but as far as we know it was not properly detailed anywhere.

(and not NP-complete languages as in [GGSW13]). We point out that our construction achieve statistical soundness (i.e., against any adversary) and therefore also satisfies (up to this additional gap and the language parameters) adaptive soundness as defined in [BH15].

Concretely, here is our construction. Assuming an SPHF on the language $\bar{\mathscr{L}}_{\mathsf{lpar}}$, we can build a witness encryption as follows:

- $\mathsf{Encrypt}_{\mathsf{WE}}(1^n, \chi, M)$ outputs $C = (\mathsf{hp}, \mathsf{H} \oplus M)$, by running $\mathsf{HashKG}(\mathsf{lpar})$, $\mathsf{ProjKG}(\mathsf{hk}, \mathsf{lpar}, \chi)$, $\mathsf{Hash}(\mathsf{hk}, \mathsf{lpar}, \chi)$ to compute $\mathsf{hk}, \mathsf{hp}, \mathsf{H}$.
- $\mathsf{Decrypt}_{\mathsf{WE}}(C, w)$ recovers $M = P \oplus \mathsf{pH}$ by parsing C as hp, P, and computing $\mathsf{pH} = \mathsf{ProjHash}(\mathsf{hp}, \mathsf{lpar}, \chi, w)$.

Theorem 5.2. *The above construction is a correct and statistically sound witness encryption scheme.*

Proof. Under the correctness of the underlying SPHF, one obtains:

$$\Pr\left[\mathsf{Decrypt}_{\mathsf{WE}}(\mathsf{Encrypt}_{\mathsf{WE}}(1^n, \chi, M), w) = M\right] \geq 1 - \mathrm{negl}(n) \ .$$

It is interesting to note, that in case of an ϵ-approximate SPHF, one can still achieve an ϵ-approximate correctness for the encryption.

The smoothness of the SPHF, ensures that for χ not in the language, H is seemingly random from the point of view of an adversary, hence $\mathsf{H} \oplus M$ is too, which guarantees the desired soundness. □

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A CCA2 and tag-CCA Security

In this section, we remind the definitions of IND-CCA2 and tag-IND-CCA2 encryption schemes, recall the generic transformation from the latter to the former, before proving tag-IND-CCA2 security for the scheme of Section 2.2.

A.1 Definitions

Definition A.1 (Labeled Encryption Scheme). *A (labeled) public-key encryption scheme is defined by four algorithms:*

- $\mathsf{KeyGen}(1^n)$ *takes as input a unary representation of the security parameter and generates a pair of keys* $(\mathsf{dk}, \mathsf{ek})$*, where* dk *is the secret decryption key and* ek *is the public encryption key;*
- $\mathsf{Encrypt}(\mathsf{ek}, \mathsf{label}, M; \rho)$ *produces a ciphertext* C *on the input message* M *under the label* label *and encryption key* ek*, using the random coins* ρ*;*
- $\mathsf{Decrypt}(\mathsf{dk}, \mathsf{label}, C)$ *outputs the plaintext* M *encrypted in* C *under the label* label*, or* $\perp$*;*

and satisfies the following property:

- ***Correctness.*** *For any security parameter* n*, with overwhelming probability over* $(\mathsf{dk}, \mathsf{ek}) \leftarrow \mathsf{KeyGen}(1^n)$*, for any label* label*, for any message* M*, for any ciphertext* $C \leftarrow \mathsf{Encrypt}(\mathsf{ek}, \mathsf{label}, M; \rho)$*, we have* $\mathsf{Decrypt}(\mathsf{dk}, \mathsf{label}, C) = M$*.*

Definition A.2 (IND-CCA2 Security). *An encryption scheme* $\mathcal{E} = (\mathsf{KeyGen}, \mathsf{Encrypt}, \mathsf{Decrypt})$ *is* IND-CCA2 *if the advantage of any polynomial-time adversary* $\mathcal{A}$ *in distinguishing* $\mathsf{Exp}^{\mathsf{cca}-0}_{\mathcal{E},\mathcal{A}}(1^n)$ *from* $\mathsf{Exp}^{\mathsf{cca}-1}_{\mathcal{E},\mathcal{A}}(1^n)$ *is negligible in the security parameter* n*, where the experiments* $\mathsf{Exp}^{\mathsf{cca}-b}_{\mathcal{E},\mathcal{A}}(1^n)$ *are depicted in Fig. 6. Informally, this notion states that an adversary should not be able to efficiently guess which message has been encrypted even if he chooses the two original plaintexts, and can ask several decryption of ciphertexts as long as they are not the challenge one.*

$\mathsf{Exp}^{\mathsf{cca}-b}_{\mathcal{E},\mathcal{A}}(1^n)$
1. $(\mathsf{dk}, \mathsf{ek}) \leftarrow \mathsf{KeyGen}(1^n)$
2. $(M_0, M_1) \leftarrow \mathcal{A}(\texttt{FIND} : \mathsf{ek}, \mathsf{ODecrypt}(\mathsf{dk}, \cdot, \cdot))$
3. $C^* \leftarrow \mathsf{Encrypt}(\mathsf{ek}, \mathsf{label}^*, M_b)$
4. $b' \leftarrow \mathcal{A}(\texttt{GUESS} : C^*, \mathsf{label}^*, \mathsf{ODecrypt}(\mathsf{dk}, \cdot, \cdot))$
5. `IF` $(\mathsf{label}^*, C^*) \in \mathcal{CT}$ `RETURN` 0
6. `ELSE RETURN` b'

$\mathsf{ODecrypt}(\mathsf{dk}, \mathsf{label}, C)$
7. Add (label, C) to $\mathcal{CT}$
8. `RETURN` $\mathsf{Decrypt}(\mathsf{dk}, \mathsf{label}, C)$

Fig. 6: Security Experiment for CCA2 security.

This IND-CCA2 notion can be relaxed into a weaker tag-IND-CCA2 security notion.

Definition A.3 (Tag-IND-CCA2 Security). *An encryption scheme* $\mathcal{E} = (\mathsf{KeyGen}, \mathsf{Encrypt}, \mathsf{Decrypt})$ *is* tag-CCA2-secure *if the advantage of any polynomial-time adversary* $\mathcal{A}$ *in distinguishing* $\mathsf{Exp}^{\mathsf{tag\text{-}cca}-0}_{\mathcal{E},\mathcal{A}}(1^n)$ *from* $\mathsf{Exp}^{\mathsf{tag\text{-}cca}-1}_{\mathcal{E},\mathcal{A}}(1^n)$ *is negligible in the security parameter* n*, where the experiments*

$\mathsf{Exp}^{\mathsf{tag\text{-}cca}-b}_{\mathcal{E},\mathcal{A}}(1^n)$ *are defined as the experiments* $\mathsf{Exp}^{\mathsf{cca}-b}_{\mathcal{E},\mathcal{A}}(1^n)$ *depicted in Fig. 6, except that the line* 5 *is superseded by:*
6. `IF` $(\mathsf{label}^*, \cdot) \in \mathcal{CT}$ `RETURN` *0.*
In other words, the adversary is not allowed to query the decryption oracle on a ciphertext with the same label label *(also called a tag and denoted* u *in this context) as the challenge one.*

Finally, we recall that the weaker IND-CPA security notion is defined similarly as the IND-CCA2 or tag-IND-CCA2 security notion, except that the adversary is not given access to the decryption oracle $\mathsf{ODecrypt}$. If the tag of a tag-IND-CCA2 encryption scheme is fixed to some public constant, then the resulting scheme is IND-CPA.

A.2 From Tag-IND-CCA2 to IND-CCA2

We can convert a tag-IND-CCA2 encryption scheme $(\mathsf{KeyGen}', \mathsf{Encrypt}', \mathsf{Decrypt}')$ with message space $\{0,1\}$ and label (a.k.a., tag) space $\{0,1\}^n$ into an IND-CCA2 encryption scheme $(\mathsf{KeyGen}, \mathsf{Encrypt}, \mathsf{Decrypt})$ with message space $\{0,1\}^\nu$ (for some ν polynomial in n) and label space $\{0,1\}^*$, using [DDN03]. Concretely, we suppose that we have a strongly unforgeable one-time signature scheme and we define:

- $\mathsf{KeyGen}(1^n)$ outputs $(\mathsf{dk}, \mathsf{ek}) \leftarrow \mathsf{KeyGen}'(1^n)$;
- $\mathsf{Encrypt}(\mathsf{ek}, \mathsf{label} \in \{0,1\}^*, M \in \{0,1\}^\nu)$ generates a signature key sk and an associated verification key pk (for the strongly unforgeable one-time signature, we suppose that pk can be represented as a n-bit string without loss of generality), computes for $1 \leq i \leq \nu$, $C_i \leftarrow \mathsf{Encrypt}'(\mathsf{ek}, \mathsf{pk}, M_i)$, and outputs $C := (C_1, \ldots, C_\nu, \mathsf{pk}, \sigma)$, where σ is a signature under sk of $(C_1, \ldots, C_\nu, \mathsf{pk}, \mathsf{label})$;
- $\mathsf{Decrypt}(\mathsf{dk}, \mathsf{label} \in \{0,1\}^*, C)$ parses C as $(C_1, \ldots, C_\nu, \mathsf{pk}, \sigma)$, abort (i.e., return $\perp$) if σ is not a valid signature of $(C_1, \ldots, C_\nu, \mathsf{pk}, \mathsf{label})$ under pk, otherwise computes for $1 \leq i \leq \nu$, $M_i = \mathsf{Decrypt}'(\mathsf{dk}, \mathsf{pk}, C_i)$, and output the bit string $M \in \{0,1\}^\nu$ corresponding to the concatenation of $M_1, \ldots, M_\nu$.

A.3 Proof of Tag-IND-CCA2 Security of our Encryption Scheme (Theorem 2.9)

The proof follows closely the proof of the original scheme in [MP12]. We proceed with Hybrid games.

Hybrid H_0. The first hybrid game H_0 is the tag-IND-CCA2 game described in Fig. 6.

Hybrid H_1*, Setup.* In a second game H_1, we pick $u^* \in \mathcal{U}$, and we set the public key to be $\boldsymbol{A}_0 = [\bar{\boldsymbol{A}} \; ; \; \boldsymbol{R}\bar{\boldsymbol{A}} - \boldsymbol{G}\; h(u^*)]$, where $(\boldsymbol{T}, \boldsymbol{A}) \leftarrow \mathsf{TrapGen}(1^n, 1^m, q)$, with $\boldsymbol{T} = [-\boldsymbol{R} \,|\, \boldsymbol{I}]$ and $\boldsymbol{A} = [\bar{\boldsymbol{A}}\,; \boldsymbol{R}\bar{\boldsymbol{A}}]$. Note that $\boldsymbol{A}_0$ is statistically close to uniform, so that this new public key is statistically indistinguishable from the one from H_0.

Hybrid H_1*, decryption queries.* To handle decryption queries on tags $u \neq u^*$, the reduction simply outputs

$$\begin{cases} \mu & \text{if } g^{-1}_{\boldsymbol{A}_0}(\boldsymbol{T}, 2\boldsymbol{c}, h(u-u^*)) = 2\boldsymbol{e} + (0,\ldots,0,\mu) \text{ where } \boldsymbol{e} \in \mathbb{Z}^m \\ & \quad \text{and } \|\boldsymbol{e}\| \leq B' \text{ with } B' := q/\Theta(\sqrt{m}) \quad , \\ \perp & \text{otherwise.} \end{cases}$$

By the correctness of the $g^{-1}_{\boldsymbol{A}_0}$ algorithm (Lemma 2.8), this procedure outputs μ if and only if $d(\boldsymbol{c} - \mathsf{Encode}(\mu), \Lambda(\boldsymbol{A}_u)) < B'$, which is exactly the same behavior than in game H_0.

Hybrid H_1, challenge ciphertext. For the challenge ciphertext, choose $\mu \in \{0,1\}$, and set tag $u = u^*$. Choose $s \in \mathbb{Z}_q^n$, $\boldsymbol{e} \leftarrow D_{\mathbb{Z},\sigma}^m$, and set $\bar{\boldsymbol{b}} = \bar{\boldsymbol{A}}\boldsymbol{s} + \boldsymbol{e}$. Define $\boldsymbol{Q} = [\boldsymbol{I}_{\bar{m}} \ ; \ \boldsymbol{R}]$. Note that $\boldsymbol{Q}\bar{\boldsymbol{b}} = \boldsymbol{A}_{u^*}\boldsymbol{s} + \boldsymbol{Q}\boldsymbol{e}$. We then set the ciphertext to be:

$$\boldsymbol{c} = \boldsymbol{Q}\bar{\boldsymbol{b}} + \hat{\boldsymbol{e}} + \mathsf{Encode}(\mu) \ ,$$

where $\hat{\boldsymbol{e}} \leftarrow D_{\mathbb{Z},\sqrt{\Sigma}}^m$ and $\Sigma = t^2\boldsymbol{I}_m - \sigma^2\boldsymbol{Q}\boldsymbol{Q}^t$.[18] We note that

$$\boldsymbol{c} = \boldsymbol{A}_{u^*}\boldsymbol{s} + \boldsymbol{e}' + \mathsf{Encode}(\mu) \ , \quad \text{where } \boldsymbol{e}' = \boldsymbol{Q}\boldsymbol{e} + \hat{\boldsymbol{e}} \ .$$

We will argue that $\boldsymbol{c}$ is distributed as in game H_0. For this, it suffices to show that $\boldsymbol{e}'$ is negligibly close to $D_{\mathbb{Z}^m,t}$. Because $\boldsymbol{Q}\boldsymbol{e}$ belongs to $\mathbb{Z}_m$ the distribution $\boldsymbol{Q}\boldsymbol{e}' + D_{\mathbb{Z}^m,\sqrt{\Sigma}}$ of $\boldsymbol{e}'$ is equal to $D_{\mathbb{Z}^m,t,\boldsymbol{Q}\boldsymbol{e}}$. It remains to apply the convolution Theorem of Peikert [Pei10, Theorem 3.1], as already detailed in [MP12, Section 5.4].

Hybrid H_2. In a third game H_2, we only change the challenge ciphertext, and we now pick $\bar{\boldsymbol{b}}$ uniformly random in $\mathbb{Z}_q^{\bar{m}}$, which is indistinguishable from the previous game by the assumption of hardness of the $\mathrm{LWE}_{\chi,q}$ problem, for $\chi = D_{\mathbb{Z},\sigma}$.

In H_2, the adversary receives $\boldsymbol{c} = \boldsymbol{Q}\boldsymbol{b} + \hat{\boldsymbol{e}} + \mathsf{Encode}(\mu)$, with $\boldsymbol{Q}\bar{\boldsymbol{b}} = [\bar{\boldsymbol{b}} \ ; \ \boldsymbol{R}\bar{\boldsymbol{b}}]^t$. But $(\bar{\boldsymbol{A}}, \boldsymbol{R}\bar{\boldsymbol{A}}, \bar{\boldsymbol{b}}, \boldsymbol{R}\bar{\boldsymbol{b}})$ is statistically negligibly close to uniform over the randomness of $\boldsymbol{R} \leftarrow D^{nk\times\bar{m}}$ by the leftover hash lemma. In particular, $\boldsymbol{c}$ is uniform and independent from the public key $\boldsymbol{A}_0$ and the message μ, so the advantage of the adversary is negligible against game H_2. □

B SPHF

In this appendix, we formally define approximate KV-SPHFs and describe the generic transformations of SPHFs sketched in Section 2.3 and summarized in Fig. 1.

B.1 Formal Definition of Approximate KV-SPHF

Definition B.1. *An approximate KV-SPHF is defined as in Definition 2.13 except that the algorithm* ProjKG *does not take as input the word $\mathcal{X}$, approximate correctness is modified accordingly, and smoothness is replaced by the following stronger property:*
(KV-)smoothness. *For any positive integer n, if* $(\mathsf{ltrap}, \mathsf{lpar}) \leftarrow \mathsf{Setup.lpar}(1^n)$*, with overwhelming probability over the randomness of* Setup.lpar*, for all f onto $\mathcal{X} \setminus \mathscr{L}_{\mathsf{lpar}}$ the following distributions have statistical distance negligible in n:*

$$\{(\mathsf{lpar}, f(\mathsf{hp}), \mathsf{hp}, \mathsf{H}) \mid \mathsf{hk} \leftarrow \mathsf{HashKG}(\mathsf{lpar}),\ \mathsf{H} \leftarrow \mathsf{Hash}(\mathsf{hk}, \mathsf{lpar}, f(\mathsf{hp})),\ \mathsf{hp} = \mathsf{ProjKG}(\mathsf{hk}, \mathsf{lpar})\} \ ,$$
$$\{(\mathsf{lpar}, f(\mathsf{hp}), \mathsf{hp}, \mathsf{H}) \mid \mathsf{hk} \leftarrow \mathsf{HashKG}(\mathsf{lpar}),\ \mathsf{H} \leftarrow \{0,1\}^{\nu},\ \mathsf{hp} = \mathsf{ProjKG}(\mathsf{hk}, \mathsf{lpar})\} \ .$$

An approximate KV-SPHF is called a KV-SPHF if it is $\epsilon(n)$-correct with $\epsilon(n)$ negligible in the security parameter n.

B.2 Generic Transformations of Bit-SPHFs and SPHFs

From Approximate Bit-SPHF to Approximate SPHF. This transformation is straightforward, we simply need to enhance the output of the hash function, by sampling several independent hash keys hk, and concatenating the output of all the corresponding Hash results.

Lemma B.2. *Let* $(\mathsf{HashKG}', \mathsf{ProjKG}', \mathsf{Hash}', \mathsf{ProjHash}')$ *be an ϵ-correct approximate bit-SPHF. Then the SPHF* $(\mathsf{HashKG}, \mathsf{ProjKG}, \mathsf{Hash}, \mathsf{ProjHash})$ *defined as follows is an $(\epsilon + \epsilon')$-correct approximate SPHF, for any constant $\epsilon' > 0$.*

[18] The procedure to sample from such a distribution is described in [Pei10, MP12].

- $\mathsf{HashKG}(\mathsf{lpar})$ *generates a hashing key* $\mathsf{hk} = (\mathsf{hk}_1, \ldots, \mathsf{hk}_\nu)$ *by running* ν *times* $\mathsf{HashKG}'(\mathsf{lpar})$*, where* $\nu = \Omega(n)$*;*
- $\mathsf{ProjKG}(\mathsf{hk}, \mathsf{lpar}, \mathcal{X})$ *derives a projection key* hp *from the hashing key* hk*, by computing* $\mathsf{hp}_i = \mathsf{ProjKG}'(\mathsf{hk}_i, \mathsf{lpar}, \mathcal{X})$ *(for* $i \in \{1, \ldots, \nu\}$*) and setting* $\mathsf{hp} = (\mathsf{hp}_1, \ldots, \mathsf{hp}_\nu)$*.*
- $\mathsf{Hash}(\mathsf{hk}, \mathsf{lpar}, \mathcal{X})$ *outputs a hash value* $\mathsf{H} \in \{0,1\}^\nu$*, by computing the various hash values* $\mathsf{H}_i = \mathsf{Hash}(\mathsf{hk}_i, \mathsf{lpar}, \mathcal{X})$ *(for* $i \in \{1, \ldots, \nu\}$*) and concatenating the ouputs:* $\mathsf{H} = \mathsf{H}_1 \| \ldots \| \mathsf{H}_\nu$*;*
- $\mathsf{ProjHash}(\mathsf{hp}, \mathsf{lpar}, \mathcal{X}, w)$ *outputs a projected hash value* $\mathsf{pH} \in \{0,1\}^\nu$*, by computing the projected hash values* $\mathsf{pH}_i = \mathsf{ProjHash}'(\mathsf{hp}_i, \mathsf{lpar}, \mathcal{X}, w)$ *(for* $i \in \{1, \ldots, \nu\}$*) and concatenating them:* $\mathsf{pH} = \mathsf{pH}_1 \| \ldots \| \mathsf{pH}_\nu$*;*

Proof. **Approximate correctness.** We have for every i:

$$\Pr_{\mathsf{hk}_i}[\mathsf{Hash}'(\mathsf{hk}_i, \mathsf{lpar}, \mathcal{X}) = \mathsf{ProjHash}'(\mathsf{hp}_i, \mathsf{lpar}, \mathcal{X}, w)] \geq 1 - \epsilon \ .$$

Hence, the property on the concatenation, using the Hoeffding bound.
Smoothness. This follows from a classical hybrid argument by considering intermediate distributions Δ_i where the first i values H_i are random, and the others are honestly computed, as each SPHF is independent and smooth. □

From Approximate Correctness to Correctness. There exists a generic transformation, implicit in [KV09], from an approximate SPHF to an SPHF. The idea is quite simple, it requires the use of an error correcting code (noted ECC in the following) capable of correcting an ϵ-fraction of errors.

Lemma B.3. *Let* $(\mathsf{HashKG}', \mathsf{ProjKG}', \mathsf{Hash}', \mathsf{ProjHash}')$ *be an* ϵ*-correct approximate SPHF (with hash values in* $\{0,1\}^\nu$*) and* ECC *be an error correcting code capable of correcting an* ϵ*-fraction of error. Then the SPHF* $(\mathsf{HashKG}, \mathsf{ProjKG}, \mathsf{Hash}, \mathsf{ProjHash})$ *defined as follows is a (regular) SPHF:*

- $\mathsf{HashKG}(\mathsf{lpar})$ *sets* $\mathsf{hk}_1 \leftarrow \mathsf{HashKG}'(\mathsf{lpar})$*, and picks a random* hk_2 *from* $\{0,1\}^\nu$*. It then returns* $\mathsf{hk} = (\mathsf{hk}_1, \mathsf{hk}_2)$*;*
- $\mathsf{ProjKG}(\mathsf{hk}, \mathsf{lpar}, \mathcal{X})$ *computes* $\mathsf{hp}_1 \leftarrow \mathsf{ProjKG}'(\mathsf{hk}_1, \mathsf{lpar}, \mathcal{X})$*, and computes* $c = \mathsf{ECC}(\mathsf{hk}_2), \mathsf{H}' \leftarrow \mathsf{Hash}'(\mathsf{hk}_1, \mathsf{lpar}, \mathcal{X})$*, and sets* $\mathsf{hp}_2 = c \oplus \mathsf{H}'$*;*
- $\mathsf{Hash}(\mathsf{hk}, \mathsf{lpar}, \mathcal{X})$ *simply outputs* $\mathsf{H} = \mathsf{hk}_2$*;*
- $\mathsf{ProjHash}(\mathsf{hp}, \mathsf{lpar}, \mathcal{X}, w)$ *computes* $\mathsf{pH}' = \mathsf{ProjHash}'(\mathsf{hp}_1, \mathsf{lpar}, \mathcal{X}, w)$ *and sets* $\mathsf{pH} = \mathsf{ECC}^{-1}(\mathsf{pH}' \oplus \mathsf{hp}_2)$*.*

We stress that this transformation always gives a SPHF (and not a KV-SPHF), even if the original approximate SPHF is an approximate KV-SPHF, as the ProjKG algorithm requires to run the approximate Hash' algorithm, and therefore requires the knowledge of the word $\mathcal{X}$.

Proof. **Approximate-correctness.** In an honest execution, the approximate correctness guarantees that $\mathsf{HW}(\mathsf{pH}', \mathsf{H}') \leq \epsilon \cdot n$. In particular, this means that $\mathsf{HW}(\mathsf{pH}' \oplus \mathsf{hp}_2, c) \leq \epsilon \cdot n$. Now, the capacity of the error-correcting code leads to the conclusion: $\mathsf{pH} = \mathsf{H}$
Smoothness. Smoothness of the original SPHF ensures that when $\mathcal{X} \notin \mathscr{L}$, H' is negligibly close to uniform even when knowing hp'. Therefore, it completely masks c (in hp_2) and thus $\mathsf{H} = \mathsf{hk}_2$ is negligibly close to uniform even when knowing $\mathsf{hp}_1 = \mathsf{hp}'$ and $\mathsf{hp}_2 = c \oplus \mathsf{H}'$. □

From Imperfectly Universal Bit-KV-PHFs to KV-SPHFs. The idea is quite simple: we first XOR the hash values of several independent executions of the bit-KV-PHF to amplify universality and get a statistically universal bit-KV-PHF. To convert the resulting bit-KV-PHF into a KV-SPHF, we then increase the output length using basic concatenation and parallel executions as in Lemma B.2.

Lemma B.4. *Let* $(\mathsf{HashKG}', \mathsf{ProjKG}', \mathsf{Hash}', \mathsf{ProjHash}')$ *be a* ϵ*-universal bit-KV-PHF. Then the SPHF* $(\mathsf{HashKG}, \mathsf{ProjKG}, \mathsf{Hash}, \mathsf{ProjHash})$ *defined as follows is a KV-SPHF:*

- $\mathsf{HashKG}(\mathsf{lpar})$ *generates a hashing key* $\mathsf{hk} = (\mathsf{hk}_{(1,1)}, \ldots, \mathsf{hk}_{(\eta,\nu)})$ *by running* $\eta \cdot \nu$ *times the original hashing key generation* $\mathsf{HashKG}'(\mathsf{lpar})$*, where* $\eta = \omega(-\log n / \log \epsilon)$ *and* ν *is the output length of the SPHF;*
- $\mathsf{ProjKG}(\mathsf{hk}, \mathsf{lpar})$ *derives a projection key* hp *from the hashing key* hk*, by computing* $\mathsf{hp}_{(i,j)} = \mathsf{ProjKG}'(\mathsf{hk}_{(i,j)}, \mathsf{lpar})$ *and setting* $\mathsf{hp} = (\mathsf{hp}_{(1,1)}, \ldots, \mathsf{hp}_{(\eta,\nu)})$*.*
- $\mathsf{Hash}(\mathsf{hk}, \mathsf{lpar}, \mathcal{X})$ *outputs a hash value* $\mathsf{H} \in \{0,1\}^{\nu}$*, by computing the various hash values* $\mathsf{H}_{(i,j)} = \mathsf{Hash}(\mathsf{hk}_{(i,j)}, \mathsf{lpar}, \mathcal{X})$*, and then* $\mathsf{H}_j = \mathsf{H}_{1,j} \oplus \cdots \oplus \mathsf{H}_{\eta,j}$ *(for* $i, j \in \{1, \ldots, \nu\}$*), and concatenating the ouputs:* $\mathsf{H} = \mathsf{H}_1 \| \ldots \| \mathsf{H}_{\nu}$*;*
- $\mathsf{ProjHash}(\mathsf{hp}, \mathsf{lpar}, \mathcal{X}, w)$ *outputs a projected hash value* $\mathsf{pH} \in \{0,1\}^{\nu}$*, by computing the hash values* $\mathsf{H}_{(i,j)} = \mathsf{Hash}(\mathsf{hk}_{(i,j)}, \mathsf{lpar}, \mathcal{X})$*, and then* $\mathsf{H}_j = \mathsf{H}_{1,j} \oplus \cdots \oplus \mathsf{H}_{\eta,j}$ *(for* $i, j \in \{1, \ldots, \nu\}$*), and concatenating the ouputs:* $\mathsf{H} = \mathsf{H}_1 \| \ldots \| \mathsf{H}_{\nu}$*;*

Proof. **Correctness.** Correctness is straightforward as the original bit-KV-PHF is statistically correct.

Smoothness. With overwhelming probability over lpar, for each $j \in \{1, \ldots, \nu\}$, for any projection key hp, we have:

$$
\begin{aligned}
&\left| 2 \cdot \Pr_{\mathsf{hk}} \left[\mathsf{H}_{1,j} \oplus \cdots \oplus \mathsf{H}_{\eta,j} = 1 \;\middle|\; \forall i \in \{1, \ldots, \eta\},\ \mathsf{hp}_{i,j} = \mathsf{ProjKG}(\mathsf{hk}_{i,j}, \mathsf{lpar}) \right] - 1 \right| \\
&= \left| \mathop{\mathbb{E}}_{\mathsf{hk}} \left[(-1)^{\mathsf{H}_{1,j}} \cdot (-1)^{\mathsf{H}_{2,j}} \cdots (-1)^{\mathsf{H}_{\eta,j}} \;\middle|\; \forall i \in \{1, \ldots, \eta\},\ \mathsf{hp}_{i,j} = \mathsf{ProjKG}(\mathsf{hk}_{i,j}, \mathsf{lpar}) \right] \right| \\
&= \left| \mathop{\mathbb{E}}_{\mathsf{hk}_{1,j}} \left[(-1)^{\mathsf{H}_{1,j}} \;\middle|\; \mathsf{hp}_{1,j} = \mathsf{ProjKG}(\mathsf{hk}_{1,j}, \mathsf{lpar}) \right] \cdots \mathop{\mathbb{E}}_{\mathsf{hk}_{\eta,j}} \left[(-1)^{\mathsf{H}_{\eta,j}} \;\middle|\; \mathsf{hp}_{\eta,j} = \mathsf{ProjKG}(\mathsf{hk}_{\eta,j}, \mathsf{lpar}) \right] \right| \\
&\leq \epsilon^{\eta} = 2^{-\omega(\log n)} \ ,
\end{aligned}
$$

where $\mathsf{H}_{(i,j)} = \mathsf{Hash}(\mathsf{hk}_{(i,j)}, \mathsf{lpar}, \mathcal{X})$, $\mathbb{E}$ denotes the expectation, and the second equality comes from the independence of the hashing keys $\mathsf{hk}_{i,j}$. In other words, if $\nu = 1$, then we would have constructed a statistically universal bit-KV-PHF.

Smoothness follows immediately. □